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Second Differential Effects of Abnormal Forces and Non-Standard Initial Conditions on Trajectories of Projectiles

. . . By . . .

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A Report
to the Office of Naval Research
upon an investigation conducted under
ONR Project Order No. 54723

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Chapter I

INTRODUCTION

1. Background of the Problem.

From the moment a projectile leaves the muzzle of the gun which launches it on its flight "down range", it is subjected to a complicated variety of forces. The study of the effects of these forces upon where the projectile goes, and upon how long it takes to get there, has for centuries fascinated mathematicians and physicists alike.

Modern guns are rifled, and hence impart to the projectile a spinning motion which affects its "stability" and thus influences its flight. But, even if one is willing to defer the tempting study of the gyroscopic properties of a spinning shell, on the ground that they are secondary in importance to the motion of its center of mass, one finds that, to compute a trajectory of the mass center, one must solve a differential system involving a number of functions which depend upon the place at which the projectile happens to be as well as upon the time when it happens to be there, not to mention the speed at which it happens to be moving.

For example, the air resistance which a bullet encounters is dependent not only upon the projectile's

speed relative to the air but also upon the atmospheric density and temperature, and these latter quantities vary from place to place and from instant to instant as the shell traverses its path. The air-speed of the projectile is itself dependent upon the speed and direction of the wind, and winds are notoriously "contrary".

At this writing, not even with the aid of high-speed computing machinery has it been practicable to obtain a complete tabulation of solutions of the differential equations of motion, covering all possible sets of initial conditions, wind-structures, density and temperature distributions, etc. The device which ballisticians have employed to overcome this difficulty is that of integrating the equations under certain relatively simple "standard", or "normal" conditions (such as zero wind, a non-rotating earth, etc.), and then computing a <u>linear approximation</u> to the <u>actual effect</u> (on range, on time-of-flight, or on some other trajectory "element") due to the "<u>disturbances</u>" (departures from "normal" conditions).

2. First Differential Effects.

The mathematical theory underlying the computation of these linear approximations, or "<u>differential effects</u>" as they are usually called, was developed by G. A. Bliss while he was at the Aberdeen Proving Ground during World War I. Some of his results were published in refer-

ences [1] and [2] of the bibliography at the end of this paper; others appear in references [3], [4], and [5]. Other investigators have provided modifications and improvements, both in the theory and in its applications. The most recent such contribution is a paper by E. J. McShane, entitled "The Differentials of Certain Functionals in Exterior Ballistics", and soon to appear in the Duke Mathematical Journal. In this paper, which will hereafter be designated as [McS], McShane shows that, for most trajectories, the "norm" (or measure) of the disturbance as used by Bliss in his proofs can be replaced by a simpler, more natural "norm".

In order to make clear the distinction between the norm used by Bliss and that used by McShane, we shall quote below an excerpt from the introduction to the paper [McS], the manuscript of which Professor McShane very kindly loaned the writer in advance of its publication.

"To be specific, let us restrict our attention to the effect of wind on range. The 'normal' condition, under which trajectories are computed, is zero wind. The range is a functional of the wind. If the wind is supposed to be a continuous function of altitude alone, we are considering a real-valued functional on a space of continuous functions. In this space we can introduce a familiar norm, the norm of a continuous function w(.) being defined to be the least upper bound of its absolute value.

Then the classical definition of a (first) differential can be applied; it is a linear functional of w(.) which differs from the change of range ΔX due to w(.) by a quantity whose ratio to norm w(.) approaches zero with norm w(.). But the existence of a differential in this sense has not heretofore been established. To the best of my knowledge, the only adequate mathematical discussion of differential effects is due to G. A. Bliss. Bliss, however, does not use the norm mentioned above. Instead, he defines the norm of w(.) to be the greater of the least upper bound of |w| and the least upper bound of the absolute value of its derivative with respect to altitude y; that is, he uses the norm of the Banach space C' instead of the norm of the space C. Now with certain disturbances this would be quite unobjectionable. For instance, the Coriolis force on a projectile is a disturbance whose value and rates of change with respect to position and velocity are computable, and the inclusion of the rates of change in the definition of the norm produces no annoyance. But the wind is an experimentally determined function, and the experiment furnishes only the mean values between a finite number of successive altitudes. The experimental result may be expected to differ from the exact wind by an error whose norm in the space C (i.e., maximum absolute value) is small; but we have no assurance that the rate of change of wind with altitude does not at some places differ widely from the experimental estimates. In order that the

mathematical treatment of differential effects shall be applicable to the experimental methods involved, it is of great importance that the norm be determined by the magnitude of $|w|$ alone, and not depend on the derivative w'. . . . . . . We have seen that disturbances can be classified into two aggregates; for those of one class it is harmless to admit the derivatives in the definition of the norm; the others, specifically wind, temperature, and density, are to be regarded as functions of altitude alone, and their derivatives are to be excluded from the definition of the norm. It is possible to avoid the annoyance of considering two separate classes of disturbances, and at the same time to gain slightly in generality, by defining the norm of a disturbing function to be the greatest of the maxima of the absolute values of the function itself and of its partial derivatives with respect to all variables <u>except</u> the altitude y. . . . . . . The norm being defined, ... with given 'normal' conditions and given disturbances, ... , do departures from 'normal' conditions produce differentiable effects? We find that they do. . . . . . . Is the difference between differential effect and actual change an infinitesimal of second order in the norm? We find that it is if the trajectory is everywhere ascending or everywhere descending; otherwise it may not be as small as second order."

3. Need For A New Theory. Summary of Results.

For most purposes, the use of differential effects in place of actual effects of disturbances has produced quite satisfactory results. Recently, however, evidence has accumulated to support the belief that linear approximations to effects of certain disturbances (e.g., a wind; or the combined presence of wind and abnormal temperature) do not give sufficiently accurate estimates of the actual effects. Because of the need of improved estimates, and because the literature contains no mathematical theory of second-order effects of disturbances, Professor McShane suggested to the writer that he undertake the development of such a theory.

The present paper consists of a portion of the results of that investigation. It includes: (1) a proof of the existence of a second differential of the mapping defined by the differential equations of motion of the mass-center of a projectile; (2) a demonstration that this second differential is expressible in terms of the second variations of the coordinates of the center of mass; (3) a derivation of the system of differential equations satisfied by the second variations, a system which has the same homogeneous terms as does the classical system of first variation equations; (4) a formula for expressing a solution of the differential system for second variations in terms of double Stieltjes integrals, (5) a simple formula for calculating

the second variations of trajectory elements from the second variations of the coordinates of the mass-center; and (6) a theorem on second differentials analogous to McShane's main theorem, (11.7) of [McS], on first differentials.

The norm of disturbance employed in obtaining the first five results above is defined in terms of the absolute values of the disturbance functions and their partial derivatives of the first and second orders, with respect to all the variables except the independent one. A somewhat simpler norm is used in establishing (6).

4. Fields for Further Investigation.

It has long been known that (first) differential effects are expressible in terms of (single) Stieltjes integrals. (By "Stieltjes integral"—both the single and the double varieties—we mean the one often referred to as the "Riemann-Stieltjes" integral, not the "Lebesgue-Stieltjes" integral.) This fact has made possible the use of a "ballistic wind"--a fictitious constant wind which has the same differential effect on range as does the actual range-wind $w_x(y)$. (A "ballistic density" and a "ballistic temperature" are similarly defined.) The ballistic wind is computed in the field from observed wind-data by the use of so-called "wind weighting-factor functions" which have been pre-computed at the ballistics laboratory. Each wind

weighting-factor function is, by definition, the ratio of the differential effect, $g(k)$, $0 \leq k \leq 1$, of a range-wind which is zero at all levels above kY (Y being the altitude of the summit of the trajectory) and +1 at all levels below kY, to the differential effect $g(1)$ of a range-wind which is one length-unit per time-unit at <u>all</u> levels. The <u>graph</u> of a weighting-factor function is known as a "weighting-factor curve", and is plotted at the laboratory. Thus the fact that (first) differential effects are expressible as (single) Stieltjes integrals has rendered possible a method whereby practically all the computations for differential effects can be performed by a group of people who can be situated far from the combat theater, and the results of their calculations can be recorded in such a manner that applications may be made in the field quickly and by means of very simple computations. (For a more detailed account of weighting factors and their applications, see references [5], [6], and [7].)

Now the results of the present investigation, which enable us to express <u>second</u> differential effects in terms of <u>double</u> Stieltjes integrals, lead us naturally to conjecture that it is possible to devise a method of weighting-factor functions for second differential effects analogous to those for first differential effects described above. But no attempt has been made to do this in the present paper. In fact, such an effort does not appear feasible until the formulas developed herein shall first

have been applied to the ballistic equations (differential equations of motion of the mass-center) themselves. This latter task is itself one of some magnitude, as will become increasingly evident in later chapters. It has been postponed as constituting an appropriate objective for a later investigation.

5. Remarks on Notation.

After experimenting with several notations, the writer concluded that, for the purpose of this paper, the most satisfactory system to use would be a slight modification of the notation employed in [McS]. Since the latter paper is not yet available in published form, it has seemed advisable (for the convenience of the reader) to quote several passages from the manuscript form of [McS]. These will be clearly indicated by quotation marks, and grateful acknowledgment is hereby made for Professor McShane's permission to make these quotations.

We shall use the summation convention with regard to repeated indices throughout the paper, except in certain parts of Chapter VI where the departure from this notation is clearly indicated.

Chapter II

THE EQUATIONS OF MOTION. A MORE GENERAL DIFFERENTIAL SYSTEM.

6. The Equations of Motion.

In this section we shall develop the differential equations of motion of the mass-center of a projectile, subject to certain simplifying assumptions, in the form in which they are usually encountered in ballistic literature. For the notational reasons indicated in § 5, we quote the following passage from [McS].

"Let us choose a rectangular coordinate system in which the direction of the earth's gravity field at (0,0,0) is along the negative y-axis. The system is supposed fixed with respect to the earth. In order to avoid ineasential annoyances, we shall select some number B and decide to confine our attention to trajectories along which the inequalities

(6.1) $|x| < B$, $|y| < B$, $|z| < B$, $|v_x| < B$, $|v_y| < B$, $|v_z| < B$,

are satisfied and whose time of transversal is less than B. The set of all points satisfying (6.1) will be named R. This restriction is not unrealistic; at any particular epoch there is an upper bound to attainable velocities, and if trajectories depart more than a few hundred miles from

the earth's surface problems of ballistics blend into those of orbit computation. The components of velocity of the center of mass of the projectile (which for brevity we shall call the components of velocity of the projectile) will be denoted by (v_x, v_y, v_z), and the length of the velocity vector by v. The components of wind-velocity will be denoted by (w_x, w_y, w_z). It is convenient, though not absolutely essential, to assume that w_y is identically zero. The velocity of the projectile with respect to the air then has components

$$(6.2) \quad u_x = v_x - w_x, \quad u_y = v_y - w_y = v_y, \quad u_z = v_z - w_z .$$

The length of the vector (u_x, u_y, u_z) we denote by u. We assume that the components of wind-velocity, the air-density ρ, and the absolute temperature ϑ are functions of position only, independent of time. We shall assume that for a projectile of given shape, size, and mass, moving through air of given chemical composition, the principal aerodynamic force has direction opposite to the vector (u_x, u_y, u_z) and magnitude muE, where m is the mass of the projectile and E is a function of u, ρ, and ϑ, and is continuous together with its partial derivatives of first and second order.* If the gravitational acceleration were of constant magnitude and direction throughout R, it would

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* For a more detailed discussion of the nature of the function E, see references [5], [6], and [7].

produce a force $(0, -mg, 0)$ on the projectile. Thus the force on the projectile has components approximately equal to $-mEu_x$, $-mEu_y-mg$, $-mEu_z$. All the other forces, whether aerodynamic or due to the change in gravity-field or Coriolis forces, we lump together and designate by the symbol (ma_x, ma_y, ma_z). For present purposes it is immaterial how these forces can be calculated; it is enough that they can be found, and that they are ordinarily considerably smaller than 'drag' mEu and gravity mg. We can now write the equations of motion in the form

$$(6.3)\quad \begin{cases} \dfrac{dv_y}{dt} = -Eu_y - g + a_y, & \dfrac{dy}{dt} = v_y, \\[2ex] \dfrac{dv_x}{dt} = -Eu_x + a_x, & \dfrac{dx}{dt} = v_x, \\[2ex] \dfrac{dv_z}{dt} = -Eu_z + a_z, & \dfrac{dz}{dt} = v_z. \end{cases}$$

Henceforth we shall assume that there is a positive number γ such that everywhere in R the inequality

$$(6.4)\qquad a_y \leqq g - \gamma$$

holds. Then at every value of t such that $u_y = v_y = 0$, we have

$$\frac{d^2y}{dt^2} = -g + a_y \leqq -\gamma .$$

This implies that whenever v_y vanishes y has a proper maximum. Then y can have no relative minimum. Between two proper maxima a minimum must occur; therefore there cannot be two times at which v_y vanishes. If on a trajectory there is a time t_s, at which $v_y = 0$, the point $(x(t_s), y(t_s), z(t_s))$ is called the summit. For $t < t_s$ we have $v_y > 0$; the corresponding arc of trajectory is the ascending branch. For $t > t_s$ we have $v_y < 0$; the corresponding arc is the descending branch. Artillery trajectories, for fire at ground targets, ordinarily have summits. Bomb trajectories in level or dive bombing have the descending branch only; anti-aircraft trajectories usually consist of an ascending branch only.

"The assumption (6.4) is a very mild one for conventional trajectories. However, by adopting it we exclude projectiles . . . which are capable of sustained level flight."

7. The Equations of Motion as a Special Case of a More General Differential System.

The system (6.3) is only one of several ways in which the equations of motion of the mass-center can be written. Other useful forms are obtained by using altitude or slope as independent variable, instead of time. In each of these forms, however, the right members of the equations are rather complicated expressions involving the

compound function E. But all these forms of the equations of motion have in common the property of being a special case of a differential system which can be written quite simply in the notation introduced in [McS]. Our investigation, therefore, will take the form of a study of this more general differential system, not only because of the greater economy of notation but also because the theory which we shall develop will then be applicable to several forms of the equations of motion instead of being confined only to the system (6.3). For this reason, we again quote from [McS], this time from § 4 of that paper.

The variable which is chosen as independent variable in the equations of motion shall be called s , and shall be assumed to vary over some interval (s_1, s_2). If s is time, this corresponds to assuming a bound on the times of flight of all projectiles considered; if, say y or x is chosen as independent variable, it follows from the assumption that the trajectories lie in a bounded part of space. The dependent variables we shall $q^1, \ldots, q^n$, and the n-uple $(q^1, \ldots, q^n)$ we shall abbreviate to q when we so desire. Superscripts affixed directly to letters are purely enumerative; exponents, when we have occasion to use them, will be affixed to parentheses. Thus $(q^3)^2$ would mean the square of the third component of the n-uple q . The dependent variables range over some open set R_o in n-space. On the set $(s_1, s_2) \times R_o$, consisting of all (n+1)-uples (s,q) with $s_1 \leq s \leq s_2$ and q in R_o , certain

functions $\varphi^1(s,q), \ldots, \varphi^m(s,q)$ are defined; these will always be assumed continuous except perhaps at a finite number of values of s, at which they may have simple jump discontinuities. We shall also assume that all the φ^α considered are less in absolute value than some uniform bound B; this corresponds to deciding in advance that we shall only consider winds, velocities, etc., which do not exceed a chosen bound. On the set in m-space defined by $|r^\alpha| \leqq B$ $(\alpha = 1,\ldots,m)$ there are n continuous functions $F^1(r),\ldots, F^n(r)$. We seek solutions of the equations

$$(7.1) \quad dq^i/ds = F^i(\varphi(s,q(s))) = F^i(\varphi^1(s,q^1,\ldots,q^n),\ldots,\varphi^m(s,q^1,\ldots,q^n)), \quad (i=1,\ldots,n)$$

with initial values

$$(7.2) \quad q^i(s_1) = q_o^i \qquad (i=1,\ldots,n)$$

lying in R_o and with $q(s)$ [i.e., $(q^1(s),\ldots,q^n(s))$] lying in R_o for $s_1 \leqq s \leqq s_2$.

"The equations (7.1) and the initial conditions (7.2) are determined when we specify the system of numbers and functions

$$(7.3) \quad (q_o^1,\ldots,q_o^n, \varphi^1,\ldots, \varphi^m, F^1,\ldots,F^n).$$

"By well-known theorems [See, e.g., E. J. McShane: Integration, (Princeton Univ. Press), p. 344.] on differential equations, for each system (7.3) with the

properties previously specified, equations (7.1), (7.2) have a solution defined on some sub-interval (s_1, s_3) of (s_1, s_2) and lying in R_o. But we cannot guarantee that the solution is unique. Systems lacking unique solutions are of no interest to us, so from the aggregate of all sets (7.3) we select the subset ϕ consisting of all (q_o, φ, F) which furnish unique solutions."

Since on the set ϕ the solutions may not be continuable to $s = s_2$, we select from ϕ a subset ϕ_o defined as follows.

(7.4) Definition: A "point" (q_o, φ, F) belongs to ϕ_o if it belongs to ϕ and the solution $q(s)$ of (7.1), (7.2) is defined and in R_o for $s_1 \leqq s \leqq s_2$.

That is, the equations (7.1), (7.2) map ϕ_o uniquely into the space C_n of n-uples of real-valued functions defined and continuous on the interval (s_1, s_2), the "image" of (q_o, φ, F) being $q(\cdot) \equiv \{q(s) \mid s_1 \leqq s \leqq s_2\}$. With reasonable norms in ϕ_o and C_n, it can be shown that this mapping is continuous on ϕ_o. In fact, this conclusion is established in Theorem (4.9) of [McS] for the norms N and N_o defined below.

(7.5) Definition: Using the summation convention, we write $|q| = (q^i q^i)^{1/2}$.

(7.6) <u>Definition</u>: In the space C_n the norm will be the customary one, $N(q(.))$, given by

$$N(q(.)) = \sup \{ |q(s)| \mid s_1 \leqq s \leqq s_2 \} .$$

(7.7) <u>Definition</u>: In ϕ_0 the norm, which we shall designate $N_0(q_0, \varphi, F)$, will be the <u>greatest</u> of the three numbers: $|q_0|$;
$\sup \{ |\varphi(s,q)| \mid s_1 \leqq s \leqq s_2, q \text{ in } R_0 \}$; and
$\sup \{ |F(r)| \mid |r^\alpha| < B, \alpha = 1,\ldots,m \}$. [By $|\varphi|$ we mean $(\varphi^\alpha \varphi^\alpha)^{1/2}$; similarly, $|F| = (F^i F^i)^{1/2}$.]

McShane's continuity theorem, on the mapping of ϕ_0 into C_n, is as follows:

(7.8) <u>Theorem</u>: When the spaces ϕ and C_n are metrized in accordance with the respective norms N_0 and N, the set ϕ_0 is an open subset of ϕ, and the mapping of ϕ_0 into C_n defined by the equations (7.1), (7.2) is continuous.

In the opening paragraph of this section, it was asserted that the system (6.3) is a special case of a more general differential system. That more general system consists, as the reader may already have inferred, of the equations (7.1) and (7.2). To see this, we identify the independent variable s with t and the dependent variables q^i with x, y, z, v_x, v_y, v_z—though not always in the order named. The functions φ^α we take to be: t, x, y, z, v_x,

v_y, v_z, w_x, w_z, ρ, θ, a_x, a_y, a_z. Thus, for the system (6.3), we have $n = 6$ and $m = 14$. Given a law of air resistance (i.e., a particular "drag function"), the function E can be calculated from the values of the 14 functions φ^{α}, and hence the right members of (6.3) are the functions F^i with the r^i replaced by the 14 functions φ^{α}, which in turn are functions of the 6 dependent variables q^i and the independent variable t.

Theoretically there is a one-to-one correspondence between projectile-types and drag functions. But the determination and tabulation of a drag function is an expensive process; not only are costly experiments (firing programs) necessary, but extensive calculations are required to reduce the data obtained from the experiments. Fortunately, the function E has as a factor not only the drag function but also the reciprocal of an empirical quantity called the <u>ballistic coefficient</u>; by altering the value of the ballistic coefficient when changing from one projectile-type to another, sufficiently similar to the first, ballisticians have found it possible to use the <u>same</u> drag function for <u>more</u> than one projectile-type. The functions F^i are always the same unless the drag function is changed; since it is rarely desired to calculate even <u>first</u> differential effects due to changes in drag function alone, we have not felt it worthwhile to complicate the notation in our theory of <u>second</u> differential effects by including terms which arise from changing the functions F^i.

Accordingly, the only "disturbances" which we shall consider in the remaining chapters are those which consist of changes in initial conditions, or of changes in the functions φ^{α}, or of both. A disturbance which consists of changes both in the initial conditions and in the functions φ^{α} we shall denote by the symbol $(\Delta q_o, \Delta \varphi)$. In accordance with this notation, $(\Delta q_o, 0)$ will denote a disturbance which consists of changes in initial conditions only; $(0, \Delta \varphi)$ will mean a disturbance which consists only of changes in the functions φ^{α}. Therefore we shall suppress the symbols F and Δ F from such notations as (q_o, φ, F) or $N_o(\Delta q_o, \Delta \varphi, \Delta F)$.

Chapter III

THE (FIRST) VARIATION EQUATIONS. CONTINUITY AND DIFFERENTIABILITY THEOREMS OF BLISS.

8. The (First) Variation Equations.

At the beginning of § 7 we made certain assumptions regarding the continuity of the functions φ^α ($\alpha = 1,\ldots,m$) and F^i ($i = 1,\ldots,n$). For our discussion of first and second differentials of the mapping defined by (7.1) and (7.2), we shall need additional assumptions concerning the differentiability of the functions φ^α and F^i. These additional properties are described in the following

(8.1) Definition: Let Φ_{ss} be that subset of Φ_o which has the properties

(8.1i) the functions F^i are twice continuously differentiable on the set of r with $|r^\alpha| \leqq B$, $\alpha = 1,\ldots,m$; and

(8.1ii) for all s in (s_1,s_2) and all q in R_o, the functions φ^α have bounded partial derivatives of the first and second orders with respect to the q^i, continuous except perhaps at a finite number of values of s.

[It should be pointed out here that the assumptions concerning second derivatives contained in (8.1i) and (8.1ii)

are unnecessarily restrictive for the theory of <u>first</u> differential effects. In fact, McShane uses in [McS] the subset ϕ_s of ϕ_0 defined by altering definition (8.1) in such a way that the word "twice" is omitted from property (8.1i) and the words "first and second orders" in property (8.1ii] are replaced by the words "first order". Since evidently $\phi_{ss} \subset \phi_s$, the results in [McS] are valid on the set ϕ_{ss}.]

We wish to emphasize that in the definition (8.1), as in the analogous definition for ϕ_s in [McS], the variable s chosen as <u>independent</u> variable is distinguished from all the others, i.e., from the variables q^i. For in (8.1ii) the partial derivatives of the functions φ^α <u>with respect to s</u> are <u>not</u> mentioned. This is the reason for using the letter s in the subscripts on the letter ϕ. For example, the set ϕ_{tt} (i.e., ϕ_{ss} for s = t) is not the same set as ϕ_{yy} (the set ϕ_{ss} when y is chosen as independent variable).

Now, let (q_0, φ) and $(\bar{q}_0, \bar{\varphi})$ be points of ϕ_{ss}, and let

$$(8.2) \quad \begin{cases} \Delta_1 q_0 \equiv \bar{q}_0 - q_0 \\ \Delta_1 \varphi(s,q) \equiv \bar{\varphi}(s,q) - \varphi(s,q) \end{cases}$$

For each real number τ, $0 \leqq \tau \leqq 1$, let

$$Q(s, \tau) \equiv (Q^1(s,\tau), \ldots, Q^n(s,\tau))$$

be the solution of the corresponding differential system

$$(8.3) \quad \begin{cases} \dfrac{dq^i}{ds} = F^i\left(\varphi(s,q) + \tau\, \Delta_1 \varphi(s,q)\right) \\[2ex] q^i(s_1, \tau) \equiv q_0^i + \tau\, \Delta_1 q_0^i \quad ; \end{cases}$$

i.e., we assume that, for each τ, $0 \leqq \tau \leqq 1$,

$$(8.4) \quad \begin{cases} \dfrac{dQ^i(s,\tau)}{ds} \equiv F^i\left(\varphi(s, Q(s,\tau)) + \tau\, \Delta_1 \varphi(s, Q(s,\tau))\right) \\[2ex] Q^i(s_1, \tau) \equiv q_0^i + \tau\, \Delta_1 q_0^i \end{cases}$$

Let

$$(8.5) \quad \begin{cases} q(s) \equiv Q(s, 0) \\ \bar{q}(s) \equiv Q(s, 1) \\ \Delta_1 q(s) \equiv \bar{q}(s) - q(s) \quad ; \end{cases}$$

and, for brevity, let

$$
(8.6)\begin{cases} F^{i}_{r^\alpha}\{\tau\} \equiv F^{i}_{r^\alpha}\left(\varphi(s, Q(s,\tau)) + \tau\, \Delta_1 \varphi(s, Q(s,\tau))\right) \\ \\ \varphi^{\alpha}_{q^j}\{\tau\} \equiv \varphi^{\alpha}_{q^j}(s, Q(s,\tau)) \end{cases}
$$

$$
[\alpha = 1, \ldots, m; \quad j = 1, \ldots, n.]
$$

with similar abbreviations for $F^{i}_{r^\alpha r^\beta}$, $\varphi^{\alpha}_{q^j q^k}$, $\Delta_1 \varphi^{\alpha}$, $\Delta_1 \varphi^{\alpha}_{q^j}$, etc. In this notation, the result of differentiating each member of (4) with respect to τ , and then interchanging the order of differentiation, may be written

$$
(8.7)\begin{cases} \dfrac{d Q^{i}_{\tau}(s,\tau)}{ds} \equiv F^{i}_{r^\alpha}\{\tau\}\left[\varphi^{\alpha}_{q^j}\{\tau\}\, Q^{j}_{\tau}(s,\tau) + \Delta_1 \varphi^{\alpha}\{\tau\} + \tau\, \Delta_1 \varphi^{\alpha}_{q^j}\{\tau\}\, Q^{j}_{\tau}(s,\tau)\right] \\ \\ Q^{i}_{\tau}(s_1, \tau) \equiv \Delta_1 q^{i}_{0} \end{cases}
$$

Thus, the functions

$$
(8.8) \qquad \eta^{i}(s) \equiv \eta^{i}(s;\, q_0, \varphi;\, \Delta_1 q_0, \Delta_1 \varphi) \equiv Q_{\tau}(s,0)
$$

satisfy

$$
(8.9)\begin{cases} \dfrac{d\eta^{i}}{ds} = F^{i}_{r^\alpha}\{0\}\left[\varphi^{\alpha}_{q^j}\{0\}\,\eta^{j} + \Delta_1 \varphi^{\alpha}\{0\}\right] \\ \\ \eta^{i}(s_1) \equiv \Delta_1 q^{i}_{0} \end{cases} ,
$$

which is the same as the system (6.11), (6.111) of [McS] with Δ replaced by Δ_1 and with $\Delta F = 0$. [$\Delta F = 0$ because we have used the <u>same</u> functions F^i in <u>both</u> points (q_0, φ, F), $(\bar{q}_0, \bar{\varphi}, F)$. In the closing paragraphs of § 7 we explained why this was done.]

The equations (8.9) have come to be known in the theory of differential equations as the "<u>equations of variation</u>".* In this paper we shall refer to them as the "<u>first variation equations</u>", to distinguish them from another system (which we shall call the "<u>second</u> variation equations")— to be deduced in Chapter IV. The functions $\eta^i(s)$ which constitute the solution of (8.9) are called the "<u>variations</u>", or, as we shall refer to them, the "<u>first</u> variations".** In fact, the (first) differential effects on trajectory elements are certain linear combinations of the (first) variations;† the first variations also are prominent in the theory of second differential effects, as we shall see in the next chapter. In applying the theory of (first) differential effects, it is seldom, if ever, necessary to solve the equations of variation (8.9). A simpler procedure (also due to Bliss) is to solve a related homogeneous differential system, said to be "adjoint" to (8.9), and

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* See [5], p. 69; [6], Chapter VIII; and [7], Chapter VIII.

**See also [9], p. 165.

† See [5], Chapter V; [6], Chapter VIII; and [7], Chapter VIII.

them to obtain the desired differential effect (linear combination of the functions η^i) by means of a fairly simple relation (equation (7) of [2] between the solutions of the system (8.9) and those of its adjoint system.

9. Linearity and Continuity of the First Variations.

In this section we shall state three results from [McS] which we shall need in the next chapter.

As observed in § 6 of [McS], the solutions of (8.1) are obviously linear in the following sense:

(9.1) Theorem: If η_1 corresponds to $(\Delta_1 q_o, \Delta_1 \varphi)$ and η_2 corresponds to $(\Delta_2 q_o, \Delta_2 \varphi)$, and if a_1 and a_2 are real numbers, then $a_1 \eta_1 + a_2 \eta_2$ corresponds to $(a_1 \Delta_1 q_o + a_2 \Delta_2 q_o,$ $a_1 \Delta_1 \varphi + a_2 \Delta_2 \varphi)$.

Moreover, the η^i are continuous functions of the disturbance $(\Delta q_o, \Delta \varphi)$ in the sense of the following

(9.2) Theorem: There exists a constant K such that

$$N(\eta(\cdot)) \leqq KN_o(\Delta q_o, \Delta \varphi).$$

The proof of this latter theorem, as given in [McS], depends upon the following lemma on differential inequalities.

(9.3) Lemma [(3.1) of [McS]] : Let A(s) be a bounded Riemann integrable (or Lebesgue summable) function, on an interval (s_1, s_2), and let B(s) be bounded and integrable (or Lebesgue summable). If f(s) is a function which satisfies the inequality

(9.4) $$f'(s) \leq A(s)f(s) + |B(s)|$$

at almost all values of s such that f(s) > 0, it also satisfies

(9.5) $$f(s) \leq \exp\int_{s_1}^{s} A(t)\,dt \left\{ |f(s_1)| + \int_{s_1}^{s} |B(t)| \left[\exp\int_{s_1}^{t} (-A(\tau))\,d\tau \right] dt \right\}.$$

We shall have occasion to use this lemma in our proof of the principal theorem in Chapter VII. McShane also employed it in his proofs of the two theorems of Bliss which are referred to in the next two sections.

10. Continuity Theorem of Bliss.

An important theorem of Bliss ([1], p. 85), concerning what he describes as "continuity of Type I", furnishes us with an estimate of the magnitude of the actual effect of a disturbance. We state it below in the form in which it is proved in [McS]. (See (5.2) of that paper.)

(10.1) Theorem (Bliss): Let (q_0, φ) be a point of ϕ_s and q(.) the corresponding solution of (7.1), (7.2). Then there is a constant K_1 such that for each

$(\bar{q}_0, \bar{\varphi})$ in Φ_0 the corresponding solution $\bar{q}(.)$ of (7.1), (7.2) satisfies the inequality

$$N(\bar{q}(.) - q(.)) \leqq K_1 N_0(\bar{q}_0 - q_0, \bar{\varphi} - \varphi).$$

11. Definition of First Differential. A New Norm. Bliss' Theorem on the First Differential.

As intimated in the last paragraph of § 8, one does not have to solve the first variation equations (8.9) in order to obtain the first differential effects on trajectory elements. The system actually to be solved is either the system adjoint to (8.9) or (see Chapter VII) the homogeneous system obtained from (8.9) by discarding its non-homogeneous terms. Nevertheless, given the equations of motion (6.3), one finds it necessary at least to write the corresponding first variation equations before either the adjoint system or the corresponding homogeneous system can be written. This can obviously be done by calculating the partial derivatives $F^i_{r^\alpha}$ and $\varphi^\alpha_{q^j}$ and then substituting the results, together with the disturbance functions $\Delta_1 \varphi^\alpha$ in the right members of the equations (8.9).

But there is still another way in which the first variation equations can be deduced from the system (6.3). Let $x = x(t)$, $y = y(t)$, $z = z(t)$ be the parametric equations of the normal trajectory defined by (6.3) with normal initial conditions and standard functions ρ, θ, $w_x = 0$,

$w_z = 0$, $a_x = 0$, $a_y = 0$, $a_z = 0$. Let $x = x(t) + \Delta x(t)$, $y = y(t) + \Delta y(t)$, $z = z(t) + \Delta z(t)$ be the parametric equations of a <u>disturbed</u> trajectory defined by (6.3) with non-standard initial conditions and/or abnormal functions $\rho + \Delta\rho$, $\theta + \Delta\theta$, w_x, w_z, a_x, a_y, a_z. Since these latter functions and E are, by the hypotheses stated in § 6, functions of x, y, z, v_x, v_y, v_z, so are the right members of (6.3). Let us expand the right members of that system as power series in $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$, $\Delta v_x(t)$, $\Delta v_y(t)$, $\Delta v_z(t)$, assumed to be convergent for sufficiently small values of these latter differences. If we retain only the <u>linear</u> terms of these series, and if from the results of this procedure we subtract the differential equations of the normal trajectory, we obtain the first variation equations with $\Delta x(t), \ldots, \Delta v_z(t)$ in the rôles of the variables η^i which appear in (8.9). (See [10], Chapter IV; [11], Chapter II; and [7], Chapter VIII.) This is the method usually employed in elementary expositions of the subject of approximations to effects of disturbances. It has certain undeniable pedagogical advantages, and undoubtedly the solutions of the first variation equations <u>are</u> "approximations"—of some kind—to the differences $\Delta x(t), \ldots, \Delta v_z(t)$.

Neither of the two procedures described above for deriving the variation equations, however, throws any light upon the important question of the accuracy of the approximations. McShane, Kelley, and Reno ([6], Chapter VIII)

call attention to the analogy between the expansion of our equations, to terms linear in the disturbances, and the directional derivatives of, say, a function of two real variables: it might be thought that if such a function has a directional derivative at the origin which is zero, regardless of the direction, then that function must be closely approximated by a constant on some region about the origin. But, as is well known, there exist functions with such directional derivatives which are discontinuous at the origin. Bliss appears to have been the first to recognize the need for an estimate for the error involved in using the variations instead of the effects; i.e., in the $s - q - \varphi$ notation, an estimate of the magnitude of the quantities $|\Delta q^i(s) - \eta^i(s)|$. He obtained an upper bound on the error by showing [see (11.3) below] that the mapping of the space ϕ_0 into the space C_n, defined by (7.1) and (7.2), has a first differential in a sense analogous to that defined by Fréchet in the theory of functions of lines.

In order to clarify the meaning of Bliss' theorem, we state next a definition of a differential of a mapping such as that just referred to. This definition paraphrases, in the notation of the present paper, one given by Hildebrandt and Graves [8], who, in turn, assign credit to Stolz, Young, and Frechet.

(11.1) <u>Definition</u>: Let $\mathfrak{F}(\psi)$ denote the mapping of ϕ_o into C defined by (7.1), (7.2), let $\psi = (q_o, \varphi)$ be a point of ϕ_o, and let $N'(\psi)$ be a norm defined in ϕ_o. Then $\mathfrak{F}$ is said to have a differential relative to N' at the point ψ if there exists a function $d\mathfrak{F}(\Delta\psi)$ on ϕ into C linear and continuous [in the sense of (9.1) and (9.2)] on ϕ, such that the function R on ϕ_o to C_n defined for $N'(\Delta\psi) \neq 0$ by

(11.1i) $$|\mathfrak{F}(\psi + \Delta\psi) - \mathfrak{F}(\psi) - d\mathfrak{F}(\Delta\psi)| = R(\psi + \Delta\psi)\, N'(\Delta\psi)$$

satisfies

(11.1ii) $$\lim_{N'(\Delta\psi) \to 0} R(\psi + \Delta\psi) = 0.$$

11. <u>A New Norm. Bliss' Theorem on the First Differential.</u>

In order to state the theorem of Bliss on the first differential of the mapping defined by (7.1), (7.2), we shall need to define a new norm. This new norm is the one denoted by the symbol $N_s(q_o, \varphi, F)$ in [McS], but we shall designate it as $N_{1s}(q_o, \varphi)$ in order to distinguish it from still another norm which we shall introduce in the next chapter. Moreover, when there is no necessity of emphasizing its arguments, we shall refer to the new norm as N_s or as N_{1s}. It is defined as follows. (See § 6 of [McS].)

(11.2) <u>Definition</u>: Let $N_{1s}(q_0, \varphi)$ denote the greatest of the three numbers:

(11.2i) $N_0(q_0, \varphi)$;

(11.2ii) $\sup |\varphi_{q^j}(s,q)|$ $(j=1,\ldots,n;\ s_1 \leqq s \leqq s_2;\ q \text{ in } R_0)$;

(11.2iii) $\sup |F_{r^\alpha}(r)|$ $(\alpha=1,\ldots,m;\ |r^\alpha| \leqq B)$.

It should be emphasized that the definition of the norm N_{1s} <u>does</u> <u>not</u> involve the partial derivatives of the functions φ^α <u>with respect to the independent variable</u> s; the derivatives of the φ^α <u>with respect to the dependent variables</u> q^j are the ones which appear in (11.2ii), It is the omission of the partial derivatives φ_s^α in the definition (11.2) which causes the norm N_{1y}(N_{1s} when s = y) to reduce to the more reasonable norm N_0 in the case of a <u>one-branched</u> trajectory (on which the choice of y as independent variable is possible) along which the disturbances being considered are assumed to be dependent on altitude alone. Such disturbances include the important ones due to winds, abnormal density, and non-standard temperature. (See § 2). For two-branched trajectories we cannot use y as independent variable. Therefore it is by no means obvious that on such trajectories the differential exists when N_{1y} is used as norm. This, however, is established in the main theorem [(11.7)] of [McS] .

In terms of the norm N_{1s} we now state Bliss' theorem on the first differential of the mapping defined by (7.1), (7.2). This is the theorem (6.5) of [McS].

(11.3) Theorem (Bliss): Let (q_0, φ) be an element of Φ_s such that the F^i have continuous partial derivatives of first and second order with respect to the r^α, and the φ^α have bounded first and second order partial derivatives with respect to the q^j, continuous except perhaps at a finite number of values of s. Then there exists a constant K_2 such that whenever $(q_0 + \Delta q_0, \varphi + \Delta\varphi)$ is in Φ_s and $\eta(.) \equiv \{\eta(s) \mid s_1 \leqq s \leqq s_2\}$ is the function defined by (8.9), the inequality

(11.3i) $N(\eta(.) - \Delta q(.)) \leqq K_2 N_0 N_{1s} \leqq K_2 [N_{1s}(\Delta q_0, \Delta\varphi)]^2$

is satisfied.

With respect to the norm N_{1s}, the first inequality in (11.3i) is a stronger conclusion than the statement that $\eta(.)$ is the differential of $q(.)$. The second inequality in (11.3i) is obviously a consequence of (11.2), since from that definition we immediately have $N_0 \leqq N_{1s}$.

Chapter IV

ON THE EXISTENCE OF THE SECOND DIFFERENTIAL. THE SECOND VARIATION EQUATIONS.

12. Meaning of the Term "Second Differential". A New Norm.

In the present chapter, it will be our objective to show the existence of the second differential of $q(\cdot)$, i.e., of the (first) differential of $\eta(\cdot)$, at (q_0, φ). But this latter term must be more clearly defined before we can proceed further. Let us first assume that $(\tilde{q}_0, \tilde{\varphi})$ belongs to Φ_{ss}, and that $(\tilde{q}_0 + \Delta_1 q_0, \tilde{\varphi} + \Delta_1 \varphi)$ belongs to Φ_0. Let $\tilde{q}(s)$ be the solution of

$$(12.1)\qquad \begin{cases} \dfrac{d\tilde{q}^i}{ds} = F^i(\tilde{\varphi}(s, \tilde{q})) \\ \tilde{q}^i(s_1) = \tilde{q}^i_0 \end{cases}$$

and let

$$(12.2)\qquad \begin{cases} \Delta_2 q(s) \equiv \tilde{q}(s) - q(s) \\ \Delta_2 \varphi(s,q) \equiv \tilde{\varphi}(s,q) - \varphi(s,q) \\ \Delta_2 q_0 \equiv \tilde{q}_0 - q_0 \end{cases}$$

Finally, let $\tilde{\eta}(.)$ be the differential of $q(.)$ at $(\tilde{q}_0, \tilde{\varphi})$ relative to the norm $N_{1s}(\Delta_1 q_0, \Delta_1 \varphi)$ i.e.,

(12.3) $\tilde{\eta}(.) \equiv \left\{ \tilde{\eta}(s) \mid s_1 \leqq s \leqq s_2 \right\}$,

in which

(12.4) $\tilde{\eta}(s) \equiv \eta(s; \tilde{q}_0, \tilde{\varphi}; \Delta_1 q_0, \Delta_1 \varphi)$

satisfies the system

$$(12.5) \quad \begin{cases} \dfrac{d\tilde{\eta}^i}{ds} = F^i_{r^\alpha}\left(\tilde{\varphi}(s, \tilde{q}(s))\right)\left[\tilde{\varphi}^\alpha_{q^j}(s, \tilde{q}(s))\,\tilde{\eta}^j + \Delta_1 \varphi^\alpha(s, \tilde{q}(s))\right] \\ \tilde{\eta}^i(s_1) \equiv \Delta_1 q^i_0 \end{cases}$$

obtained from (8.9) by replacing $\varphi(s, q(s))$ with $\tilde{\varphi}(s, \tilde{q}(s))$. It is our desire to find a function which approximates to the difference

(12.6) $\Delta\eta(s) \equiv \tilde{\eta}(s) - \eta(s)$

in a manner (to be more carefully described later) such that it may appropriately be called the differential of $\eta(.)$, or the second differential of $q(.)$. But, as we have seen in our review of the theory of the first differential, the definition of a differential requires the prior specification of the <u>norm</u> of the disturbance. As might be expected, we find it necessary to introduce a new norm—in

general larger than N_{1s}—in order to establish the existence of a second differential of the mapping defined by (7.1),(7.2). This new norm we shall denote by $N_{2s}(q_o, \varphi)$ and define as follows:

(12.7) $N_{2s}(q_o, \varphi)$ = the greater of the numbers:

(i) $N_{1s}(q_o, \varphi)$;

(ii) $\sup \{ |\varphi_{qj}\, q^k(s,q)| \}$,

($j,k = 1,\ldots,n$; $s_1 \leq s \leq s_2$;

q in R_o).

We observe at once that N_{2s} shares an important property with the norm N_{1s}—on trajectories along which it is possible to use altitude y as independent variable of disturbances due to wind, abnormal temperature, and abnormal density, the norm N_{2s} reduces to the simpler, more natural norm N_o. And now we can formulate our definition of the second differential of q(.), relative to the norm N_{2s}.

(12.8) Definition: For fixed $\Delta_1 \psi$, the first differential $d\mathcal{F}(\psi, \Delta_1 \psi)$ depends on ψ and, relative to some norm $N^*(\psi)$ defined in Φ_o, may have a differential which is a function of ψ and $\Delta_2 \psi$, linear and continuous [as in (9.1), (9.2)] in $\Delta_2 \psi$. If so, we denote it by the symbol $d^2\mathcal{F}(\psi, \Delta_1 \psi, \Delta_2 \psi)$ and call it the second differential of $\mathcal{F}$ at ψ relative to the norm N^*.

From the definition (11.1) of a first differential, the function $R'(\psi, \Delta_1 \psi, \Delta_2 \psi)$ defined by

$$|d\mathcal{F}(\psi + \Delta_2\psi, \Delta_1\psi) - d\mathcal{F}(\psi, \Delta_1\psi) - d^2\mathcal{F}(\psi, \Delta_1\psi, \Delta_2\psi)| = R'(\psi, \Delta_1\psi, \Delta_2\psi)\, N^*(\Delta_2\psi)$$

tends to zero as $N^*(\Delta_2\psi)$ does. We shall show more than this. It turns out that $d^2\mathcal{F}$ is bilinear and symmetric in $\Delta_1\psi$, $\Delta_2\psi$, and satisfies

$$|\mathcal{F}(\psi + \Delta\psi) - \mathcal{F}(\psi) - d\mathcal{F}(\psi, \Delta\psi) - \tfrac{1}{2} d^2\mathcal{F}(\psi, \Delta\psi, \Delta\psi)| = R''(\psi, \Delta\psi)\, N_{2s}(\Delta\psi)\, N_{2s}(\Delta\psi)$$

wherein $R''(\psi, \Delta\psi)$ tends to zero with $\Delta\psi$.

13. A First Approximation to the Second Differential.

Since our objective is to show the existence of the second differential of the mapping (7.1), (7.2) relative to the norm N_{2s}, we begin in this section our search for the function $d^2\mathcal{F}$ referred to in the definition (12.8). Our quest is not immediately successful; we find, instead, certain functions which we designate $\bar{\zeta}^i(s)$, and which are almost the ones we desire. They lack, however, the property of bilinearity in the disturbances. On the other hand, this observation enables us to alter certain steps in the argument of the present section in such a manner that, in the section which follows, we are able to obtain the functions $\zeta^i(s)$ which we seek.

In order to make use of results in [McS], we shall find it convenient to write (8.9) in the form which (8.4) assumes when $\tau = 0$; i.e., in the form

$$(13.1)\quad \begin{cases} \dfrac{dq^i}{ds} = F^i\,(\varphi(s,\, q)\,) \\ q_o^i\,(s_1) = q_o^i \;, \end{cases}$$

the system satisfied by $q(s)$. With this purpose in mind, we define, for $i = 1, 2, \ldots\ldots, n$, the functions

$$(13.2)\quad G^i\,(p^1,\ldots,p^n\,) \equiv p^i \;,$$

$$(13.3)\quad \vartheta^i(s,\eta) \equiv F^i_{r\alpha}\,(\varphi(s,q(s)))\left[\varphi^\alpha_{q^j}(s,q(s))\,\eta^j + \Delta_1\,\varphi^\alpha(s,q(s))\right],$$

$$(13.4)\quad \tilde{\vartheta}^i(s,\eta) \equiv F^i_{r\alpha}(\tilde{\varphi}(s,\tilde{q}(s)))\left[\tilde{\varphi}^\alpha_{q^j}(s,\tilde{q}(s))\,\tilde{\eta}^j + \Delta_1\,\varphi^\alpha(s,\tilde{q}(s))\right],$$

$$(13.5)\quad \vartheta\,(s,\eta\,) \equiv (\,\vartheta^1(s,\eta\,),\;\ldots,\;\vartheta^n(s,\eta\,)\,)\,,$$

$$(13.6)\quad \tilde{\vartheta}\,(s,\eta) \equiv (\,\tilde{\vartheta}^1(s,\eta\,),\;\ldots,\;\tilde{\vartheta}^n(s,\eta\,)\,),$$

$$(13.7)\quad \Delta\vartheta\,(s,\eta\,) \equiv \tilde{\vartheta}\,(s,\,\eta\,) - \vartheta\,(s,\,\eta\,).$$

Then (8.9) and (12.5) become, respectively,

$$(13.8)\quad \begin{cases} \dfrac{d\eta^i}{ds} = G^i\,(\,\vartheta\,(s,\,\eta\,)\,) \\ \eta^i\,(s_1) = \Delta_1\,q_o^i \end{cases}$$

and

$$(13.9)\quad \begin{cases} \dfrac{d\,\tilde{\eta}^i}{ds} = G^i(\tilde{\vartheta}(s, \tilde{\eta})) \\ \tilde{\eta}^i(s1) = \Delta_1 q_0^{\,i} \ . \end{cases}$$

If we write

$$N_0(\Delta\vartheta) = N_0(0, \Delta\vartheta),$$
$$N_s(\Delta\vartheta) = N_{1s}(0, \Delta\vartheta),$$

and let

$$\bar{\zeta}(s) \equiv \bar{\zeta}(s, \Delta_1\varphi, \Delta_2\varphi)$$

be the function which satisfies the system

$$(13.10)\quad \begin{cases} \dfrac{d\bar{\zeta}^i}{ds} = G^i_{\varphi^\nu}\big(\vartheta(s,\eta(s))\big)\Big[\vartheta^\nu_{\eta^j}(s,\eta(s))\,\bar{\zeta}^j + \Delta\vartheta^\nu(s,\eta(s))\Big] \\ \bar{\zeta}^i(s_1) \equiv \tilde{\eta}(s_1) - \eta(s_1) \equiv 0, \qquad (\nu = 1, \ldots, n) \ , \end{cases}$$

then, by (11.8) —since the functions G^i, ϑ^i clearly satisfy the hypotheses of that theorem—it follows that

$$(13.11)\quad N(\Delta\eta(\cdot) - \bar{\zeta}(\cdot)) = O(N_0(\Delta\vartheta)\,N_s(\Delta\vartheta))$$

But the "disturbance" $\Delta\vartheta$ depends upon the "disturbances" $(\Delta_1 q_0, \Delta_1\varphi)$, $(\Delta_2 q_0, \Delta_2\varphi)$. Hence it is to the norms of these latter "disturbances"

that we wish to compare the left member of (13.11). This requires that we express $\Delta\vartheta$ in terms of functions whose magnitudes, relative to these latter norms, we can estimate by previously established results. The first step in this process is to observe that

$$\begin{aligned}(13.12)\quad \Delta\vartheta^i(s,\eta) = \eta^j \Big\{ & F^i_{r^\alpha}(\tilde\varphi(s,\tilde q(s)))\,\tilde\varphi^\alpha_{q^j}(s,\tilde q(s)) \\ & - F^i_{r^\alpha}(\varphi(s,q(s)))\,\varphi^\alpha_{q^j}(s,q(s))\Big\} \\ + \Big\{ & F^i_{r^\alpha}(\tilde\varphi(s,\tilde q(s)))\,\Delta_1\varphi^\alpha(s,\tilde q(s)) \\ & - F^i_{r^\alpha}(\varphi(s,q(s)))\,\Delta_1\varphi^\alpha(s,q(s))\Big\}.\end{aligned}$$

To evaluate the right member of (13.12), we shall find it convenient to use the abbreviations

$$(13.13)\quad \begin{cases} F^i_{r^\alpha}[\tau] \equiv F^i_{r^\alpha}\big(\varphi(s,q(s)+\tau\,\Delta_2 q(s)) + \tau\,\Delta_2\varphi(s,\tilde q(s))\big), \\ \\ \varphi^\alpha_{q^j}[\tau] \equiv \varphi^\alpha_{q^j}(s,\, q(s)+\tau\,\Delta_2 q(s)), \end{cases}$$

and similar ones for the partial derivatives $F^i_{r^\alpha r^\beta}$, $\varphi^\alpha_{q^j q^k}$, $\Delta_1\varphi^\alpha_{q^j}$, $\Delta_2\varphi^\alpha_{q^j}$, etc. In this notation,

$$(13.14)\quad \Delta_1\varphi^{\alpha}(s,\tilde{q}(s)) = \Delta_1\varphi^{\alpha}[0] + \left\{ \Delta_1\varphi^{\alpha}[1] - \Delta_1\varphi^{\alpha}[0] \right\}$$

$$= \Delta_1\varphi^{\alpha}[0] + \Delta_2 q^{j}(s)\int_0^1 \Delta_1\varphi^{\alpha}_{q^j}\,[\tau]\,d\tau\,;$$

$$(13.15)\quad F^{i}_{r\alpha}(\tilde{\varphi}(s,\tilde{q}(s))) = F^{i}_{r\alpha}[0] + \left\{ F^{i}_{r\alpha}[1] - F^{i}_{r\alpha}[0] \right\}$$

$$= F^{i}_{r\alpha}[0] + \int_0^1 \frac{d}{d\tau}F^{i}_{r\alpha}[\tau]\,d\tau$$

$$= F^{i}_{r\alpha}[0] + \Delta_2 q^{k}(s)\int_0^1 F^{i}_{r\alpha\,r\beta}[\tau]\cdot\varphi^{\beta}_{q^k}[\tau]\,d\tau$$

$$+ \Delta_2\varphi^{\beta}(s,\tilde{q}(s))\int_0^1 F^{i}_{r\alpha\,r\beta}[\tau]\,d\tau$$

$$= F^{i}_{r\alpha}[0] + \Delta_2 q^{k}(s)\int_0^1 F^{i}_{r\alpha\,r\beta}[\tau]\,\varphi^{\beta}_{q^k}[\tau]\,d\tau$$

$$+ \Delta_2\varphi^{\beta}[0]\int_0^1 F^{i}_{r\alpha\,r\beta}[\tau]\,d\tau$$

$$+ \Delta_2 q^{k}(s)\int_0^1 F^{i}_{r\alpha\,r\beta}[\tau]\,d\tau\int_0^1 \Delta_2\varphi^{\beta}_{q^k}[\tau]\,d\tau;$$

and

$$(13.16) \quad \tilde{\varphi}^{\alpha}_{q^j}(s,\tilde{q}(s)) = \varphi^{\alpha}_{q^j}(s,\tilde{q}(s)) + \Delta_2\varphi^{\alpha}_{q^j}(s,\tilde{q}(s))$$

$$= \varphi^{\alpha}_{q^j}[0] + \int_0^1 \frac{d}{d\tau}\varphi^{\alpha}_{q^j}[\tau]\,d\tau$$

$$+ \Delta_2\varphi^{\alpha}_{q^j}[0] + \int_0^1 \frac{d}{d\tau}\Delta_2\varphi^{\alpha}_{q^j}[\tau]\,d\tau$$

$$= \varphi^{\alpha}_{q^j}[0] + \Delta_2 q^k(s)\int_0^1 \varphi^{\alpha}_{q^j q^k}[\tau]\,d\tau$$

$$+\Delta_2\varphi^{\alpha}_{q^j}[0] + \Delta_2 q^k(s)\int_0^1 \Delta_2\varphi^{\alpha}_{q^j q^k}[\tau]\,d\tau \quad .$$

Substituting (13.14), (13.15), and (13.16) in (13.12) yields

$$(13.17)\quad \Delta\theta^{i}(s,\eta)=\eta^{j}\Big\{\Delta_2 q^{K}(s)F^{i}_{r^\alpha}[0]\int_0^1 \varphi^{\alpha}_{q^j q^K}[\tau]\,d\tau$$

$$+F^{i}_{r^\alpha}[0]\,\Delta_2\varphi^{\alpha}_{q^j}[0]$$

$$+\Delta_2 q^{K}(s)F^{i}_{r^\alpha}[0]\int_0^1 \Delta_2\varphi^{\alpha}_{q^j q^K}[\tau]\,d\tau$$

$$+\Delta_2 q^{K}(s)\varphi^{\alpha}_{q^j}[0]\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\varphi^{\beta}_{q^K}[\tau]\,d\tau$$

$$+(\Delta_2 q^{K}(s))^2\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\varphi^{\beta}_{q^K}[\tau]\,d\tau\int_0 \varphi^{\alpha}_{q^j q^K}[\tau]\,d\tau$$

$$+\Delta_2 q^{K}(s)\Delta_2\varphi^{\alpha}_{q^j}[0]\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\varphi^{\beta}_{q^K}[\tau]\,d\tau$$

$$+(\Delta_2 q^{K}(s))^2\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\varphi^{\beta}_{q^K}[\tau]\,d\tau\int_0^1 \Delta_2\varphi^{\alpha}_{q^j q^K}[\tau]\,d\tau$$

$$+\Delta_2\varphi^{\beta}[0]\,\varphi^{\alpha}_{q^j}[0]\int_0 F^{i}_{r^\alpha r^\beta}[\tau]\,d\tau$$

$$+\Delta_2 q^{K}(s)\Delta_2\varphi^{\beta}[0]\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \varphi^{\alpha}_{q^j q^K}[\tau]\,d\tau$$

$$+\Delta_2\varphi^{\beta}[0]\,\Delta_2\varphi^{\alpha}_{q^j}[0]\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\,d\tau$$

$$+\Delta_2\varphi^{\beta}[0]\Delta_2 q^{K}(s)\int_0^1 F^{i}_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \Delta_2\varphi^{\alpha}_{q^j q^K}[\tau]\,d\tau$$

$$+ \Delta_2 q^{\kappa}(s)\varphi^{\alpha}_{q^j}[0]\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \Delta_2\,\varphi^{\beta}_{q^\kappa}[\tau]\,d\tau$$

$$+(\Delta_2 q^{\kappa}(s))^2\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \Delta_2\varphi^{\beta}_{q^\kappa}[\tau]\,d\tau\int_0^1 \varphi^{\beta}_{q^j q^\kappa}[\tau]\,d\tau$$

$$+\Delta_2 q^{\kappa}(s)\Delta_2\varphi^{\alpha}_{q^j}[0]\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \Delta_2\,\varphi^{\beta}_{q^\kappa}[\tau]\,d\tau$$

$$+(\Delta_2 q^{\kappa}(s))^2\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \Delta_2\varphi^{\beta}_{q^\kappa}[\tau]\,d\tau\int_0^1 \Delta_2\varphi^{\alpha}_{q^j q^\kappa}[\tau]\,d\tau\Big\}$$

$$+ \Delta_2\, q^{j}(s)F^i_{r^\alpha}[0]\int_0^1 \Delta_1\varphi^{\alpha}_{q^j}\,[\tau]\,d\tau$$

$$+ \Delta_2 q^{\kappa}(s)\,\Delta_1\varphi^{\alpha}[0]\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,\varphi^{\beta}_{q^\kappa}[\tau]\,d\tau$$

$$+ \Delta_2 q^{j}(s)\Delta_2 q^{\kappa}(s)\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,\varphi^{\beta}_{q^\kappa}[\tau]\,d\tau\int_0^1 \Delta_1\varphi^{\alpha}_{q^j}[\tau]\,d\tau$$

$$+ \Delta_1\varphi^{\alpha}[0]\,\Delta_2\varphi^{\beta}[0]\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau$$

$$+ \Delta_2 q^{j}(s)\,\Delta_2\varphi^{\beta}[0]\int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau\int_0^1 \Delta_1\varphi^{\alpha}_{q^j}[\tau]\,d\tau$$

$$+ \Delta_2 q^k(s) \Delta_1 \varphi^\alpha[0] \int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau \int_0^1 \Delta_2 \varphi^\beta_{q^k}[\tau]\,d\tau$$

$$+ \Delta_2 \dot{q}^j(s) \Delta_2 q^k(s) \int_0^1 \Delta_1 \varphi^\alpha_{\dot{q}^j}[\tau]\,d\tau \int_0^1 F^i_{r^\alpha r^\beta}[\tau]\,d\tau \int_0^1 \Delta_2 \varphi^\beta_{q^k}[\tau]\,d\tau.$$

Since we assume that $\Delta F = 0$, and since we shall need to distinguish between $N_{1s}(\Delta_1 q_0, \Delta_1 \varphi)$ and $N_{1s}(\Delta_2 q_0, \Delta_2 \varphi)$, we shall find it helpful to abbreviate these last named norms by $N_{1s}(\Delta_1)$ and $N_{1s}(\Delta_2)$, respectively, as well as to employ similar abbreviations for other norms.

There are twenty-two terms in the right member of (13.17). To avoid writing each one again, let us denote them — after replacing $\dot{\eta}^j$ by $\dot{\eta}^j(s)$ — by $T_1, \ldots, T_{22}$, respectively, the subscript indicating the position of the term in the order of its occurrence in the right member of (13.17).

We recall: first, that by (10.1) we have

(13.18) $\Delta_2 q^k(s) = O(N_0(\Delta_2))$;

second, that by the continuity of the functions $\dot{\eta}^j(s)$ [see (9.2)], we have

(13.19) $\eta^i(s) = O(N_o(\Delta_1))$;

and, third, that the first and second partial derivatives of and F^i are, by (8.1), bounded. Hence we conclude that

(13.20.1) $T_1 = O(N_o(\Delta_1)\, N_o(\Delta_2))$

(13.20.2) $T_2 = O(N_o(\Delta_1)\, N_{1s}(\Delta_2))$

(13.20.3) $T_3 = O(N_o(\Delta_1)\, N_o(\Delta_2)\, N_{2s}(\Delta_2))$

(13.20.4) $T_4 = O(N_o(\Delta_1)\, N_o(\Delta_2))$

(13.20.5) $T_5 = O(N_o(\Delta_1)\, N_o(\Delta_2)^2)$

(13.20.6) $T_6 = O(N_o(\Delta_1)\, N_o(\Delta_2)\, N_{1s}(\Delta_2))$

(13.20.7) $T_7 = O(N_o(\Delta_1)\, N_o(\Delta_2)^2\, N_{2s}(\Delta_2))$

(13.20.8) $T_8 = O(N_o(\Delta_1)\, N_o(\Delta_2))$

(13.20.9) $T_9 = O(N_o(\Delta_1)\, N_o(\Delta_2)^2)$

(13.20.10) $T_{10} = O(N_o(\Delta_1)\, N_o(\Delta_2)\, N_{1s}(\Delta_2))$

(13.20.11) $T_{11} = O(N_o(\Delta_1)\, N_o(\Delta_2)^2\, N_{2s}(\Delta_2))$

(13.20.12) $T_{12} = O(N_o(\Delta_1)\, N_o(\Delta_2)\, N_{1s}(\Delta_2))$

(13.20.13) $T_{13} = O(N_o(\Delta_1)\, N_o(\Delta_2)^2\, N_{1s}(\Delta_2))$

(13.20.14) $T_{14} = O(N_o(\Delta_1)\, N_o(\Delta_2)\, N_{1s}(\Delta_2)^2)$

(13.20.15) $T_{15} = O(N_o(\Delta_1)\, N_o(\Delta_2)^2\, N_{1s}(\Delta_2)\, N_{2s}(\Delta_2))$

(13.20.16) $T_{16} = O(N_o(\Delta_2) N_{1s}(\Delta_1))$

(13.20.17) $T_{17} = O(N_o(\Delta_2) N_o(\Delta_1))$

(13.20.18) $T_{18} = O(N_o(\Delta_2)^2 N_{1s}(\Delta_1))$

(13.20.19) $T_{19} = O(N_o(\Delta_1) N_o(\Delta_2))$

(13.20.20) $T_{20} = O(N_o(\Delta_2)^2 N_{1s}(\Delta_1))$

(13.20.21) $T_{21} = O(N_o(\Delta_2) N_o(\Delta_1) N_{1s}(\Delta_2))$

(13.20.22) $T_{22} = O(N_o(\Delta_2)^2 N_{1s}(\Delta_1) N_{1s}(\Delta_2))$

Consequently, from (13.17) and the equations (13.20.1) to (13.20.22), inclusive, we see that

$$N_o(\Delta\vartheta) = \mathcal{O}(N_{1s}(\Delta_1) N_{2s}(\Delta_2))$$

and

$$N_{1s}(\Delta\vartheta) \equiv N_s(\Delta\vartheta) = \mathcal{O}(N_{2s}(\Delta_2)).$$

Hence (13.11) becomes

(13.21) $N(\Delta\varphi(.) - \overline{\zeta}(.)) = \mathcal{O}(N_{1s}(\Delta_1) N_{2s}(\Delta_2)^2).$

REMARK: If we restrict our attention to the case in which $\Delta_2\varphi = \mu\Delta_1\varphi$, and the ratio μ is independent of q^j and also satisfies

$$\Delta_2 q_o = \mu\Delta_1 q_o,$$

as we shall later wish to do, then

$$N_{1s}(\Delta_1)\, N_{2s}(\Delta_2) = \mu\, N_{1s}(\Delta_1)\, N_{2s}(\Delta_1);$$

so

$$N_0(\Delta\vartheta) = \mu\, O(N_{1s}(\Delta_1)\, N_{2s}(\Delta_1)),$$

$$N_{1s}(\Delta\vartheta) = \mu\, O(N_{2s}(\Delta_1)),$$

and

$$(13.22)\quad N(\Delta\eta(.) - \bar{\zeta}(.)) = \mu^2 O(N_{1s}(\Delta_1)\, N_{2s}(\Delta_1)^2).$$

Now, because of (13.21) or (13.22), we might be led to conjecture that the function $\bar{\zeta}(.)$ defined by (13.10) is the function we are looking for, viz., the differential of $\eta(.)$. But a moment's reflection is sufficient to convince us that this is not the case. In fact, $\bar{\zeta}(.)$ lacks one of the essential properties of a differential: it fails to be linear in the disturbance $(\Delta_2 q_0, \Delta_2 \varphi)$ to which the difference $\Delta\eta(.)$ is attributable. To see this we need only to observe, from equations (13.20.1) through (13.20.22) inclusive, that $\Delta\vartheta(s, \eta(s))$ is not linear in $(\Delta_2 q_0, \Delta_2 \varphi)$, and, consequently, that $\bar{\zeta}(.)$ is also not linear in this disturbance. On the other hand, $\Delta\vartheta(s, \eta(s))$ is linear in the disturbance $(\Delta_1 q_0, \Delta_1 \varphi)$. Moreover, since both $\eta(.)$ and $\tilde{\eta}(.)$ are (first) differentials of q(.) due to the disturbance $(\Delta_1 q_0, \Delta_1 \varphi)$ and hence are linear in that disturbance, it follows that $\Delta\eta(.) = \tilde{\eta}(.) - \eta(.)$ is at most linear

in $(\Delta_1 q_o, \Delta_1 \varphi)$. These considerations suggest that, in our search for the (first) differential of $\eta(.)$, we replace the term $\Delta\vartheta^\nu(s, \eta(s))$ in (13.10) by an approximation thereto which is bilinear in the two disturbances $(\Delta_1 q_o, \Delta_1 \varphi)$ and $(\Delta_2 q_o, \Delta_2 \varphi)$. Such an approximation is furnished by the sum of those terms in $\Delta\vartheta^\nu(s, \eta(s))$ which are linear in $(\Delta_2 q_o, \Delta_2 \varphi)$. From equation (13.17) we see that these terms are T_1, T_2, T_4, T_8, T_{16}, T_{17}, and T_{19}. Their sum, however, is still not quite the approximation we desire, for two reasons: first, we have no means whereby to evaluate the integrals in these terms; and, second, ultimately we shall need to consider the case for which $\Delta_2 q_o \equiv \Delta_1 q_o$ and $\Delta_2\varphi \equiv \Delta_1\varphi$, under which circumstances the terms T_1, T_4, T_{16}, and T_{17} contain the very functions $\Delta_1 q^k(s)$ whose computation we wish to avoid!

14. The Second Differential.

But the difficulties mentioned in the last paragraph of § 13 are not insurmountable. We may overcome the first, in fact, by the simple device of replacing each integral in T_1, T_4, T_8, T_{16}, T_{17}, T_{19} by its integrand evaluated at $\tau = 0$. For, by the mean value theorem of the integral calculus, each such integral is equal to its integrand evaluated at $\tau = 0$ plus a term or terms of at least the first degree in the disturbance $(\Delta_2 q_o, \Delta_2 \varphi)$.

The second difficulty, too, is easily circumvented. We need only define

$$\eta_2(s) \equiv \eta(s;\ q_0, \varphi\ ;\ \Delta_2 q_0, \Delta_2 \varphi\ )$$

as the solution of the system

$$(14.1)\quad \begin{cases} \dfrac{d\eta_2^i}{ds} = F^i_{r\alpha}(\varphi(s,q(s)))\left[\varphi^\alpha_{q^j}(s,q(s)) + \Delta_2 \varphi^\alpha(s,q(s))\right] \\ \eta_2^i(s_1) \equiv \Delta_2 q_0^i \ , \end{cases}$$

so that $\eta_2(s)$ is identical with $\eta(s)$ when $\Delta_2 \equiv \Delta_1$, and, by the continuity of the functions $\eta^j_2(s)$,

(14.2) $\eta^j_2(s) = \mathcal{O}(N_0(\Delta_2))$.

Then we may replace $\Delta_2 q(s)$ by $\eta_2(s)$ in T_1, T_4, T_{16}, T_{17}, without disturbing the estimates contained in the right members of the corresponding equations (13.20.1), (13.20.4), (13.20.16), and (13.20.17).

Let $\overline{T}_1$, $\overline{T}_2$, $\overline{T}_4$, $\overline{T}_8$, $\overline{T}_{16}$, $\overline{T}_{17}$, $\overline{T}_{19}$ denote the results of revising the corresponding terms $T_1, \ldots, T_{19}$ in the manner described in the preceding paragraph. Then the corresponding equations in the sequence (13.20.1), ..., (13.20.19) are still valid when T_1, T_2, T_4, T_8, T_{16}, T_{17}, T_{19}, are replaced, respectively, by $\overline{T}_1$, $\overline{T}_2$, $\overline{T}_4$, $\overline{T}_8$, $\overline{T}_{16}$, $\overline{T}_{17}$, $\overline{T}_{19}$. Let

$$d\,\vartheta^i(s, \eta(s); \Delta_1\varphi\ , \Delta_2\varphi\ )$$

denote the sum

$$\overline{T}_1 + \overline{T}_2 + \overline{T}_4 + \overline{T}_8 + \overline{T}_{16} + \overline{T}_{17} + \overline{T}_{19} \,;$$

i.e.,

(14.3) $d\vartheta^i(s, \eta(s); \Delta_1\varphi, \Delta_2\varphi)$

$$= F^i_{r\alpha}(\varphi(s\, q(s)))\, \varphi^\alpha_{q^j q^k}(s, q(s))\, \eta^j(s)\, \eta_2^k(s)$$

$$+ F^i_{r\alpha}(\varphi(s, q(s)))\, \Delta_2 \varphi^\alpha_{q^j}(s, q(s))\, \eta^j(s)$$

$$+ F^i_{r^\alpha r^\beta}(\varphi(s, q(s)))\, \varphi^\alpha_{q^j}(s, q(s))\, \varphi^\beta_{q^k}(s, q(s))\, \eta^j(s)\, \eta_2^k(s)$$

$$+ F^i_{r^\alpha r^\beta}(\varphi(s, q(s)))\, \varphi^\alpha_{q^j}(s, q(s))\, \Delta_2\varphi^\beta(s, q(s))\, \eta^j(s)$$

$$+ F^i_{r\alpha}(\varphi(s, q(s)))\, \Delta_1 \varphi^\alpha_{q^j}(s, q(s))\, \eta_2^j(s)$$

$$+ F^i_{r^\alpha r^\beta}(\varphi(s, q(s)))\, \varphi^\beta_{q^k}(s, q(s))\, \Delta_1\varphi^\alpha(s, q(s))\, \eta_2^k(s)$$

$$+ F^i_{r^\alpha r^\beta}(\varphi(s, q(s)))\, \Delta_1\varphi^\alpha(s, q(s))\, \Delta_2\varphi^\beta(s, q(s)) \,.$$

Now let $\zeta(s; \Delta_1\varphi, \Delta_2\varphi)$ be the function defined by the system

$$(14.4)\begin{cases} \dfrac{d\zeta^i}{ds} = G^i_{p^\nu}(\vartheta(s,\eta(s)))\left[\vartheta^\nu_{\eta^j}(s,\eta(s))\zeta^j + d\vartheta^\nu(s,\eta(s);\Delta_1\varphi,\Delta_2\varphi)\right] \\ \zeta^i(s_1;\Delta_1\varphi,\Delta_2\varphi) \equiv \tilde{\eta}(s_1) - \eta(s_1) \equiv 0 \end{cases},$$

and let us compare the function $\zeta(s;\Delta_1\varphi,\Delta_2\varphi)$ with the function $\bar{\zeta}(s;\Delta_1\varphi,\Delta_2\varphi)$ defined by (13.10). We can estimate their difference by Theorem (10.1). To this end, let

$$H^i(u^1,\ldots,u^n) = u^i,$$

$$\psi^i(s,\zeta) \equiv G^i_{p^\nu}(\vartheta(s,\eta(s)))\left[\vartheta^\nu_{\eta^j}(s,\eta(s))\zeta^j + d\vartheta^\nu(s,\eta(s);\Delta_1\varphi,\Delta_2\varphi)\right]$$

and

$$\bar{\psi}^i(s,\zeta) \equiv G^i_{p^\nu}(\vartheta(s,\eta(s)))\left[\vartheta^\nu_{\eta^j}(s,\eta(s))\zeta^j + \Delta\vartheta^\nu(s,\eta(s))\right].$$

Then (13.10) and (14.4) become, respectively,

$$(14.5)\begin{cases} \dfrac{d\bar{\zeta}^i}{ds} = H^i(\bar{\psi}(s,\bar{\zeta})) \\ \bar{\zeta}^i(s_1) = 0, \end{cases}$$

and

$$(14.6)\quad \begin{cases} \dfrac{d\zeta^i}{ds} = I^i(\psi(s,\zeta)) \\[2ex] \zeta^i(s_1) = 0, \end{cases}$$

and by (10.1), we have

$$(14.7)\quad N\left(\bar{\zeta}(\cdot;\Delta_1\varphi,\Delta_2\varphi) - \zeta(\cdot;\Delta_1\varphi,\Delta_2\varphi)\right) = O\left(N_0\left(\bar{\psi}(s,\zeta) - \psi(s,\zeta)\right)\right).$$

But

$$(14.8)\quad \bar{\psi}^i(s,\zeta) - \psi^i(s,\zeta) = \Delta\vartheta^i(s,\eta(s)) - d\vartheta^i(s,\eta(s);\Delta_1\varphi,\Delta_2\varphi);$$

and, since from (11.3) it follows that

$$N(\eta_2(\cdot) - \Delta_2 q(\cdot)) = O(N_0(\Delta_2)\, N_{1s}(\Delta_2)),$$

we have from (13.17) and (14.3) the relations

$$T_1 - \bar{T}_1 = O(N_0(\Delta_1)\, N_0(\Delta_2)\, N_{1s}(\Delta_2)),$$

$$T_4 - \bar{T}_4 = O(N_0(\Delta_1)\, N_0(\Delta_2)\, N_{1s}(\Delta_2)),$$

$$T_{16} - \bar{T}_{16} = O(N_0(\Delta_2)\, N_{1s}(\Delta_2)\, N_{2s}(\Delta_1)),$$

and

$$T_{17} - \bar{T}_{17} = O(N_0(\Delta_1)\, N_0(\Delta_2)\, N_{1s}(\Delta_2)).$$

Moreover, a comparison of (13.17) and (14.3) also yields the following relations:

$$T_2 - \overline{T}_2 \equiv 0 ,$$

$$T_8 - \overline{T}_8 = \mathcal{O}(N_0(\Delta_1)\, N_0(\Delta_2)\, N_0(\Delta_2)) ,$$

$$T_{19} - \overline{T}_{19} = \mathcal{O}(N_0(\Delta_1)\, N_0(\Delta_2)\, N_0(\Delta_2)) .$$

Finally, from (14.8), the seven relations above, and the equations (13.20.1), . . ., (13.20.22), we conclude that

$$(14.9) \quad N_0(\overline{\Psi}(s,\zeta) - \psi(s,\zeta)) = O(N_{2s}(\Delta_1) N_0(\Delta_2) N_{2s}(\Delta_2)) .$$

Consequently (14.7) may be replaced by

$$(14.10) \quad N(\overline{\zeta}(\cdot\,; \Delta_1\varphi, \Delta_2\varphi) - \zeta(\cdot\,; \Delta_1\varphi, \Delta_2\varphi))$$
$$= \mathcal{O}(N_{2s}(\Delta_1)\, N_0(\Delta_2)\, N_{2s}(\Delta_2)) .$$

Hence we conclude, from (13.21) and (14.10), that

$$(14.11) \quad N(\Delta\eta(\cdot) - \zeta(\cdot\,; \Delta_1\varphi, \Delta_2\varphi))$$
$$\leqq N(\Delta\eta(\cdot) - \overline{\zeta}(\cdot\,; \Delta_1\varphi, \Delta_2\varphi))$$
$$+ N(\overline{\zeta}(\cdot\,; \Delta_1\varphi, \Delta_2\varphi) - \zeta(\cdot\,; \Delta_1\varphi, \Delta_2\varphi))$$
$$= \mathcal{O}(N_{2s}(\Delta_1)\, N_{2s}(\Delta_2)\, N_{2s}(\Delta_2)) .$$

Of course, in the case to which the remark following (13.21) applies, i.e., when

$$\Delta_2 \varphi = \mu \Delta_1 \varphi$$

and

$$\Delta_2 q_o = \mu \Delta_1 q_o,$$

with μ independent of q^j, (14.10) may be replaced by

$$(14.12) \quad N\left(\zeta(\cdot;\Delta_1\varphi,\mu\Delta_1\varphi) - \zeta(\cdot;\Delta_1\varphi,\mu\Delta_1\varphi)\right)$$
$$= \mu^2 C\left([N_{2s}(\Delta_1)]^2 N_0(\Delta_1)\right) ;$$

so that from (13.22) and (14.12) we have

$$(14.13) \quad N\left(\Delta\eta(\cdot) - \zeta(\cdot;\Delta_1\varphi,\mu\Delta_1\varphi)\right)$$
$$= \mu^2 C\left(N_{1s}(\Delta_1)[N_{2s}(\Delta_1)]^2\right)$$
$$= C\left(N_{1s}(\Delta_1)[N_{2s}(\mu\Delta_1)]^2\right) ,$$

instead of (14.11). Moreover, from (13.19), (14.2), and (14.3), it follows that $d\vartheta^i(s;\eta(s);\Delta_1\varphi,\Delta_2\varphi)$ is bilinear in the disturbances $(\Delta_1 q_o, \Delta_1\varphi)$ and $(\Delta_2 q_o, \Delta_2\varphi)$. Consequently, we conclude from (14.4) that $\zeta(s;\Delta_1\varphi,\Delta_2\varphi)$ also has this property.

15. An Improved Approximation for the Actual Effects of a Disturbance on the Functions $q^i(s)$.

We are now in a position to turn our attention to the problem of approximating $\Delta_1 q^i(s)$ by an expression of the form

$$\eta^i(s) + \frac{1}{2}\,\zeta^i(s;\ \Delta_1\varphi,\ \Delta_1\varphi)$$

and of obtaining an estimate of the error of such an approximation.

Let us suppose that we are given a disturbance $(\Delta q_0, \Delta\varphi)$, and let us continue to suppose that (q_0, φ) and $(q_0 + \Delta q_0,\ \varphi + \Delta\varphi)$ are in ϕ_{ss}. Then, as a convenience in what follows, and better to emphasize the dependence upon q_0, φ of the solution of

$$(15.1)\qquad \begin{cases} \dfrac{dq^i}{ds} = F^i(\varphi(s,q)) \\ q^i(s_1) = q_0^i\ , \end{cases}$$

<u>let us denote that solution by the symbol</u>

$$q(s;\ q_0, \varphi).$$

Because $\Delta F \equiv 0$ we do notinclude F in the symbol. Now let ν be any positive integer. It is easily verified that, for $j = 1, 2, \ldots, \nu - 1$, the points

$$(q_0 + \frac{j}{\nu} \Delta q_0, \quad \varphi + \frac{j}{\nu} \Delta \varphi)$$

are in ϕ_{ss}. From (8.8) and (11.3) it follows that, for each $i = 1, \ldots, n$, and $j = 1, \ldots, \nu$,

$$(15.2) \quad q^i(s, q_0 + \frac{j}{\nu} \Delta q_0, \varphi + \frac{j}{\nu} \Delta \varphi) - q^i(s; q_0 + \frac{j-1}{\nu} \Delta q_0, \varphi + \frac{j-1}{\nu} \Delta \varphi)$$

$$= \eta^i(s; q_0 + \frac{j-1}{\nu} \Delta q_0, \varphi + \frac{j-1}{\nu} \Delta \varphi; \frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi)$$

$$+ O\left(N_0\left(\frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi\right) N_{1s}\left(\frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi\right)\right).$$

But from (14.13) we conclude that

$$(15.3) \quad \eta^i(s, q_0 + \frac{j-1}{\nu} \Delta q_0, \varphi + \frac{j-1}{\nu} \Delta \varphi; \frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi)$$

$$= \eta^i(s, q_0, \varphi; \frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi) + \zeta^i(s; \frac{1}{\nu} \Delta \varphi, \frac{j-1}{\nu} \Delta \varphi)$$

$$+ O\left(N_{1s}\left(\frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi\right) \left[N_{2s}\left(\frac{j-1}{\nu} \Delta q_0, \frac{j-1}{\nu} \Delta \varphi\right)\right]^2\right)$$

Moreover

$$(15.4) \quad N_0\left(\frac{1}{\nu} \Delta q_0, \frac{1}{\nu} \Delta \varphi\right) = \frac{1}{\nu} N_0(\Delta q_0, \Delta \varphi),$$

and

$$(15.\;)\quad N_{1s}\left(\frac{1}{\nu}\,\Delta q_0, \frac{1}{\nu}\Delta\varphi\right) = \frac{1}{\nu} N_{1s}(\Delta q_0, \Delta\varphi)$$

Since $j - 1 \leqq \nu$, we also have

$$(15.6)\quad N_{2s}\left(\frac{j-1}{\nu}\,\Delta q_0, \frac{j-1}{\nu}\Delta\varphi\right) \leqq N_{2s}(\Delta q_0, \Delta\varphi)\;.$$

Hence, by combining (15.2) with (15.3), utilizing (15.4), (15.5), and (15.6), and by recalling the linearity of the η's and the bilinearity of the ζ's, we obtain

$$(15.7)\quad q^i\left(s; q_0+\frac{j}{\nu}\Delta q_0, \varphi+\frac{j}{\nu}\Delta\varphi\right) - q^i\left(s; q_0+\frac{j-1}{\nu}\Delta q_0, \varphi+\frac{j-1}{\nu}\Delta\varphi\right)$$

$$= \frac{1}{\nu}\eta^i(s; q_0, \varphi; \Delta q_0, \Delta\varphi) + \frac{j-1}{\nu^2}\zeta^i(s; \Delta\varphi, \Delta\varphi)$$

$$+ \frac{1}{\nu}\,O\left(N_{1s}(\Delta q_0, \Delta\varphi)\left[N_{2s}(\Delta q_0, \Delta\varphi)\right]^2\right)$$

$$+ \frac{1}{\nu^2}\,O\left(N_0(\Delta q_0, \Delta\varphi)\, N_{1s}(\Delta q_0, \Delta\varphi)\right)$$

Replacing j in (15.7) by 1, 2, . . . , ν, consecutively, and adding the results yields

(15.8) $q^i(s;\, q_0 + \Delta q_0,\, \varphi + \Delta\varphi) - q^i(s;\, q_0,\, \varphi)$

$$= \eta^i(s;\, q_0,\, \varphi;\, \Delta q_0,\, \Delta\varphi) + \frac{1}{\nu^2}\,\frac{\nu(\nu-1)}{2}\,\zeta^i(s,\, \Delta\varphi,\, \Delta\varphi)$$

$$+ C\left(N_{1s}(\Delta q_0,\, \Delta\varphi)\left[N_{2s}(\Delta q_0,\, \Delta\varphi)\right]^2\right)$$

$$+ \frac{1}{\nu}\, C^2\left(N_0(\Delta q_0,\, \Delta\varphi)\, N_{1s}(\Delta q_0,\, \Delta\varphi)\right)$$

Whence, upon taking limits as $\nu \rightarrow \infty$, we find that

(15.9) $q^i(s;\, q_0 + \Delta q_0,\, \varphi + \Delta\varphi) - q^i(s;\, q_0,\, \varphi)$

$$= \eta^i(s;\, q_0,\, \varphi,\, \Delta q_0,\, \Delta\varphi)$$

$$+ \frac{1}{2}\,\zeta^i(s,\, \Delta\varphi,\, \Delta\varphi)$$

$$+ C\left(N_{1s}(\Delta q_0,\, \Delta\varphi)\left[N_{2s}(\Delta q_0,\, \Delta\varphi)\right]^2\right).$$

Finally when we write

$$q^i(s) \equiv q^i(s;\, q_0,\, \varphi),$$

$$\Delta q^i(s) \equiv q^i(s;\, q_0 + \Delta q_0,\, \varphi + \Delta\varphi) - q^i(s,\, q_0,\, \varphi),$$

$$\eta^i(s) \equiv \eta^i(s;\, q_0,\, \varphi;\, \Delta q_0,\, \Delta\varphi)$$

and

$$\zeta^{1}(s) = \zeta^{1}(s;\ \Delta\varphi, \Delta\varphi),$$

it is readily seen from (15.9) that there exists a constant K_3 such that

$$(15.10)\quad N\left(\Delta q(\cdot) - \eta(\cdot) - \tfrac{1}{2}\zeta(\cdot)\right) \leqq K_3\, N_{1s}(\Delta q_0, \Delta\varphi)\left[N_{2s}(\Delta q_0, \Delta\varphi)\right]^2 .$$

Thus we have not only shown that, as $N_{2s}(\Delta q_0, \Delta\varphi)$ tends to zero, the quotient

$$\frac{N\left(\Delta q(\cdot) - \eta(\cdot) - \tfrac{1}{2}\zeta(\cdot)\right)}{\left[N_{2s}(\Delta q_0, \Delta\varphi)\right]^2}$$

also approaches zero; (15.10) is a somewhat stronger conclusion than the statement that the function

$$\zeta(\cdot) \equiv \{\zeta(s) \mid s_1 \leqq s \leqq s_2\}$$

defined by (14.4) with $\Delta_1 q_0 \equiv \Delta_2 q_0 \equiv \Delta q_0$, $\Delta_2\varphi \equiv \Delta_1\varphi \equiv \Delta\varphi$, is the <u>second differential</u> of $q(\cdot)$ <u>with respect to the norm</u> $N_{2s}(\Delta q_0, \Delta\varphi)$.

16. The Second Variations; The Second Variation Equations.

We may now observe that (14.4) can be expressed in terms of the given functions F^i, φ^α, $\Delta\varphi^\alpha$, and initial values q_0^i, Δq_0^i. For if we compute $\vartheta_{\eta^j}^i$ from (13.3), put Δq_0 in place of $\Delta_1 q_0$ and $\Delta_2 q_0$, put $\Delta\varphi$ in place of $\Delta_1\varphi$ and $\Delta_2\varphi$ in equations (3.9), (14.1), and (14.3), and if we note that $G^i_{p^\nu} = 0$ for $\nu \neq i$, $G^i_{p^i} = 1$, we find that the system (14.4) becomes

$$
(16.1)\quad \left\{
\begin{aligned}
\frac{d\zeta^i}{ds} &= F^i_{r^\alpha}(\varphi(s,q(s)))\varphi^\alpha_{q^j}(s,q(s))\,\zeta^j \\
&\quad + F^i_{r^\alpha r^\beta}(\varphi(s,q(s)))\varphi^\alpha_{q^j}(s,q(s))\varphi^\beta_{q^k}(s,q(s))\,\eta^j(s)\eta^k(s) \\
&\quad + 2F^i_{r^\alpha r^\beta}(\varphi(s,q(s)))\varphi^\alpha_{q^j}(s,q(s))\Delta\varphi^\beta(s,q(s))\,\eta^j(s) \\
&\quad + F^i_{r^\alpha r^\beta}(\varphi(s,q(s)))\Delta\varphi^\alpha(s,q(s))\Delta\varphi^\beta(s,q(s)) \\
&\quad + 2F^i_{r^\alpha}(\varphi(s,q(s)))\Delta\varphi^\alpha_{q^j}(s,q(s))\,\eta^j(s) \\
&\quad + F^i_{r^\alpha}(\varphi(s,q(s)))\varphi^\alpha_{q^j q^k}(s,q(s))\,\eta^j(s)\eta^k(s) \\
\zeta^i(s_1) &= 0.
\end{aligned}
\right.
$$

In fact, (16.1) is the system which one obtains by differentiating (8.7) with respect to τ and then setting $\tau = 0$ in the result. Thus the functions $\zeta^i(s)$ and $Q_{\tau\tau}(s,0)$ are identical. We shall call them the

"second variations" of the functions $q^i(s)$. (See [9], p. 165); The equations (16.1) which determine the second variations we shall call the "second variation equations".

If we compare (14.4) with (8.9), we see that Theorem (9.2) applied to the function $\zeta(.)$ defined by (14.4) yields the result that

$$N(\zeta(.)) \leq K\, N_0(0, d\vartheta)$$

From (14.3) we have, when $\Delta_1 = \Delta_2$,

$$N_0(0, d\vartheta) = \mathcal{O}(N_0(\Delta q_0, \Delta\varphi)\, N_{1s}(\Delta q_0, \Delta\varphi)).$$

Hence there exists a constant K^* such that

$$N(\zeta(.)) \leq K^*\, N_0(\Delta q_0, \Delta\varphi)\, N_{1s}(\Delta q_0, \Delta\varphi). \tag{16.2}$$

Chapter V

SECOND DIFFERENTIAL EFFECTS ON TRAJECTORY ELEMENTS

17. Effects on Trajectory Elements Compared With Effects on Coordinates of the Mass-Center.

Let $x(t)$, $y(t)$, $z(t)$ be the functions which, together with their derivatives $\dot{x}(t) = v_x(t)$, $\dot{y}(t) = v_y(t)$, $\dot{z}(t) = v_z(t)$, constitute the solution of the system (6.3) subject to the normal initial conditions, which include $x(0) = 0 = y(0) = z(0)$. That is, the curve

$$(17.1) \qquad \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

is the "normal trajectory". If at some time $t = T > 0$ we have $y(T) = 0$, then we call T the "time-of-flight" and $X = x(T)$ the "range". If $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$ denote the actual effects on the coordinates x, y, z, respectively, due to a given disturbance, then the curve

$$(17.2) \qquad \begin{cases} x = x(t) + \Delta x(t) \\ y = y(t) + \Delta y(t) \\ z = z(t) + \Delta z(t) \end{cases}$$

is the "<u>disturbed trajectory</u>". If T' is a positive value of t such that $y(T') + \Delta y(T') = 0$, then $X' = x(T') + \Delta x(T')$ is the range on the disturbed trajectory and

$$(17.3) \qquad \begin{aligned} \Delta X &= X' - X \\ &= x(T') + \Delta x(T') - x(T) \end{aligned}$$

is the "<u>range-effect</u>" due to the given disturbance.

We note that the range-effect ΔX is defined as the difference between the x-coordinates of a point R on the disturbed trajectory (17.2) and apoint P on the normal trajectory (17.1), the points P and R having been "matched" (made to correspond) by the requirement that they have <u>equal altitudes</u> ($y = 0$ at both P and R). Since we can <u>compute</u> ΔX by (17.3) only if we know both $\Delta x(t)$ and T', which knowledge can be acquired only by computing the disturbed trajectory (17.2), and since it is this additional computation which the ballistician wishes to avoid, we need a method of <u>approximating</u> ΔX with reasonable accuracy. Our theory of differential effects should furnish such a method.

But most trajectories are computed with time t as independent variable, so when we attempt to apply our theory of differential effects we find that, while we can with a fair amount of ease obtain a first-order approximation [one of the functions $\eta^1(t)$ defined by (8.9) for $s = t$] to

Δ $x(t)$, we obtain no immediate information about the value of T'. The fundamental difficulty is that the variation equations enable us to find approximations to the actual differences in the coordinates of two points, P on the normal trajectory and Q on the disturbed trajectory, which correspond by virtue of having equal values of <u>time</u> t; whereas we require an estimate of the difference between the x-coordinates of P and a point R (generally different from Q) which match because they have equal values of <u>another</u> variable, the <u>altitude</u> y. Similar difficulties are experienced in attempting to approximate the effects on time-of-flight, on deflection (departure from the plane of fire), and on other elements of the trajectory.

If we are to be able to apply our theory of differentials, then it is clear that we must be able to express differential effects computed at equal values of one variable in terms of differential effects computed at equal values of some other variable.

18. <u>Relation Between First Differential Effects on Trajectory Elements and First Differential Effects on Coordinates of Mass-Center.</u>

The objective outlined in the last paragraph of the preceding section was attained for first differential effects by Bliss in reference [2], through the use of certain theorems on implicit functions contained in his earlier

paper [1]. In a simpler, though perhaps less general manner, McShane, Kelley, and Reno develop a similar relation in their book [6]. It is this latter relation which we shall state in the present section, since we shall find it useful in deducing an analogous formula for <u>second</u> differential effects in § 19.

Let P be any point of a <u>normal</u> trajectory. Let A and B be any two variables which can be used as independent variables on an arc of trajectory including P. Let $\bar{\Delta} = (\Delta q_0, \Delta \varphi)$ be any disturbance. Let C be either (i) the value, at a specific point of a trajectory, of one of the fundamental quantities $q^i(s)$ defined by (7.1), (7.2) or (ii) a twice continuously differentiable function of the q^i. Let A_0, B_0, C_0 be the values of A, B, C, respectively, at the point P of the normal trajectory. We shall assume that the norm $N_{2s} = N_{2s}(\Delta q_0, \Delta \varphi)$ of the disturbance $\bar{\Delta}$ is sufficiently small to insure the existence of points Q and R on the <u>disturbed</u> trajectory such that (i) at Q the variable A has the value A_0, and (ii) at R the variable B has the value B_0. Let $\Delta B(\bar{\Delta} \mid A = A_0)$ denote the difference between the value of B at Q and the value of B at P; similarly, let $\Delta C(\bar{\Delta} \mid A = A_0)$ be the difference between the value of C at Q and the value of C at P. That is, <u>at Q</u> we have

$$(18.1)\quad A = A_0,\quad B = B_0 + \Delta B(\bar{\Delta} \mid A = A_0),$$
$$C = C_0 + \Delta C(\bar{\Delta} \mid A = A_0).$$

Likewise, we denote the values of A, B, and C <u>at</u> <u>R</u> by

$$(18.2)\quad A = A_0 + \Delta A(\bar{\Delta} \mid B = B_0),\quad B = B_0,$$
$$C = C_0 + \Delta C(\bar{\Delta} \mid B = B_0),$$

respectively. Finally, if we let $dC(\bar{\Delta} \mid A = A_0)$ denote the first differential effect of $\bar{\Delta}$ on C at $A = A_0$, and if we assign similar meanings to the symbols $dC(\bar{\Delta} \mid B = B_0)$ and $dC(\bar{\Delta} \mid B = B_0)$ and $dB(\bar{\Delta} \mid A = A_0)$, then the relation established by McShane, Kelley, and Reno ([6], Chapter VII, § 1) is

$$(18.3)\quad dC(\bar{\Delta} \mid B = B_0) = dC(\bar{\Delta} \mid A = A_0) - C'(B_0)\,dB(\bar{\Delta} \mid A = A_0),$$

in which the symbol $C'(B_0)$ means the value at $B = B_0$ of the derivative $C'(B)$ of C with respect to B <u>on the normal trajectory.</u>

For example, to obtain the first differential effect on range attributable to a disturbance $\bar{\Delta}$ (the case discussed in § 17), we apply (18.3) with $C = x$, $B = y$, $B_0 = 0$, $A = t$, $A_0 = T$, and $x'(y) = \dot{x}(t)/\dot{y}(t)$. If we denote the angle-of-fall by ω, the application of (18.3) with the substitutions just indicated yields

$$(18.4)\quad dx(\bar{\Delta} \mid y = 0) = dx(\bar{\Delta} \mid t = T) + \cot\omega \, dy(\bar{\Delta} \mid t = T)$$

$$= dx(\bar{\Delta} \mid t = T) - \frac{\dot{x}(T)}{\dot{y}(T)} \, dy(\bar{\Delta} \mid t = T)$$

as the first differential effect on range due to the disturbance $\bar{\Delta}$. If $s = t$ and the variables $q^1, \ldots, q^6$ in (7.1) are x, y, z, v_x, v_y, v_z, respectively, then the differentials $dx(\bar{\Delta} \mid t = T)$ and $dy(\bar{\Delta} \mid t = T)$ in (18.4) are the values at T of the functions $\eta^1(t)$ and $\eta^2(t)$, respectively, defined by (8.9) with $s = t$.

19. A Relation Between Second Differential Effects Computed With Two Different Independent Variables.

If we attempt to improve the estimate furnished by (18.4) for the range effect ΔX by utilizing the second variations ζ^1 defined by (16.1) when $s = t$, we face the same difficulty which confronted us in § 17: we require an (improved) estimate of the difference between the x-coordinates of the points R and P—matched because they have equal altitudes—whereas the variation equations (both first and second) furnish us only the tools for estimating the differences between the coordinates (and their time-derivatives) of the points Q and P, which correspond by virtue of having equal values of time. That is to say, if we wish to employ second differentials to improve our approximation to the range-effect ΔX, we shall have need for a formula—analogous to (18.4)—which enables us to de-

duce second differential effects computed at equal values of one variable from second and/or first differential effects computed at equal values of some other variable. It is our purpose in this section to develop such a relation.

We shall use the notation introduced in § 18, and, in addition, we shall denote by $d^2C(\bar{\Delta} \mid A = A_0)$ the second differential effect on C at $A = A_0$ attributable to the disturbance $\bar{\Delta}$; i.e., the second differential defined in (18.8) for the case in which $\Delta_1\Psi = \Delta_2\Psi = \bar{\Delta}$. We ascribe similar meanings to $d^2C(\bar{\Delta} \mid B = B_0)$ and to $d^2B(\bar{\Delta} \mid A = A_0)$, and we let $C''(B)$ denote the second derivative of C with respect to B on the normal trajectory. Then the relation which we shall establish is that

$$
\begin{aligned}
(19.1)\quad d^2C(\bar{\Delta} \mid B = B_0) &= d^2C(\bar{\Delta} \mid A = A_0) \\
&\quad - C'(B_0)\, d^2B(\bar{\Delta} \mid A = A_0) \\
&\quad + C''(B_0)\, [dB(\bar{\Delta} \mid A = A_0)]^2 \\
&\quad - 2dC'(\bar{\Delta} \mid A = A_0)\cdot dB(\bar{\Delta} \mid A = A_0).
\end{aligned}
$$

Toward this end we note first, that, if B is the independent variable, then from point Q to point R on the disturbed trajectory the change in B is $-\Delta B(\bar{\Delta} \mid A = A_0)$ while that in C is $\Delta C(\bar{\Delta} \mid B = B_0) - \Delta C(\bar{\Delta} \mid A = A_0)$. Hence by Taylor's Theorem there exists a number B1, between B_0 and $B_0 + \Delta B(\bar{\Delta} \mid A = A_0)$, such that

$$(19.2)\quad \Delta C(\bar{\Delta} \mid B = B_0) - \Delta C(\bar{\Delta} \mid A = A_0)$$
$$= - \Delta B(\bar{\Delta} \mid A = A_0)\ \bar{C}'(B_0 + \Delta B(\bar{\Delta} \mid A = A_0))$$
$$+ \frac{1}{2}\left[- \Delta B(\bar{\Delta} \mid A = A_0)\right]^2 \bar{C}''(B_1)\ ,$$

wherein the symbol $\bar{C}'(B_0 + \Delta B(\bar{\Delta} \mid A = A_0))$ means the value of the derivative of C with respect to B at the point on the disturbed trajectory at which $B = B_0 + \Delta B(\bar{\Delta} \mid A = A_0)$, with a similar interpretation for $\bar{C}''(B_1)$. From (10.1) we see that ΔC will not exceed a finite multiple of $N_0 = N_0(\bar{\Delta})$. This applies equally well to ΔB, $\Delta C'$, and $\Delta C''$. In particular, the values of $C''(B)$ at the points of normal and disturbed trajectories with $B = B_1$ differ by at most a bounded multiple of N_0. Moreover, B_1 differs from B_0 by at most a bounded multiple of N_0, so the values of $C''(B)$ at the points of the normal trajectory corresponding to $B = B_1$ and to $B = B_0$ also differ by at most a bounded multiple of N_0. Hence

$$(19.3)\quad \bar{C}''(B_1) = C''(B_0) + \epsilon_1 N_0,$$

in which ϵ_1 is bounded. By hypothesis, $C(B)$ is a twice continuously differentiable function $K(q)$ of the q^i, so the equation

$$C'(B) = K_{q^i}(q(B))\ q'^i(B)$$

expresses $C'(B)$ as a continuously differentiable function of the q^i. If B <u>can</u> be used as independent variable along the whole trajectory (and not just along an arc which includes P), then by (11.3) the q^i are differentiable functionals of the disturbance $\bar{\Delta}$, and, by a slight adaptation of a proof to be found in most advanced calculus texts (e.g., [12], p. 77; or [13], p. 72), it follows that the differential $dC'(\bar{\Delta} \mid B = B_0)$ exists. On the other hand, if B <u>cannot</u> be employed as independent variable along the whole trajectory (as is the case, for example, when B = y), then theorem (11.3) does not apply. But for the important case B = y, McShane has shown ([McS], theorem (11.7)) that even then the q^i are differentiable functionals of the disturbance. His method of proof depends upon the introduction, through a transformation of coordinates, of a new variable s capable of use as independent variable along the whole trajectory and uniquely determined by y when y does not exceed a number less than but arbitrarily near the maximum ordinate of the normal trajectory. In Chapter VII we shall describe in more detail the coordinate system used by McShane; it suffices here to say that the method is quite general and can be applied to any variable B which can be used as independent variable on an arc of trajectory including P. Thus, in either case, the existence of $dC'(\bar{\Delta} \mid B = B_0)$ is guaranteed. Of course, a similar conclusion holds concerning $dC'(\bar{\Delta} \mid A = A_0)$. By (11.31), or by (11.9) of [McS] in case B = y, we have

(19.4) $\quad \bar{C}'(B_0) = C'(B_0) + dC'(\bar{\Delta} \mid B = B_0) + \epsilon_2 N_{1s}^2,$

in which $N_{1s} = N_{1s}(\bar{\Delta})$ and ϵ_2 is bounded.

But from (15.10) it follows that

(19.5) $\quad \Delta C(\bar{\Delta} \mid A = A_0) = dC(\bar{\Delta} \mid A = A_0) + (1/2)d^2C(\bar{\Delta} \mid A = A_0) + \epsilon_3 N_{2s}^3$

and

(19.6) $\quad \Delta B(\bar{\Delta} \mid A = A_0) = dB(\bar{\Delta} \mid A = A_0) + (1/2)d^2B(\bar{\Delta} \mid A = A_0) + \epsilon_4 N_{2s}^3,$

in which ϵ_3 and ϵ_4 are bounded and $N_{2s} = N_{2s}(\bar{\Delta})$. In order to simplify the notation somewhat, throughout the remainder of the present argument we shall write ΔB, ΔC, dB, etc., for $\Delta B(\bar{\Delta} \mid A = A_0)$, $\Delta C(\bar{\Delta} \mid A = A_0)$, $dB(\bar{\Delta} \mid A = A_0)$, etc.

Using this abbreviated notation, we observe that by Taylor's Theorem

(19.7) $\quad \bar{C}'(B_0 + \Delta B) = \bar{C}'(B_0) + \Delta B \cdot \bar{C}''(B_0) + \epsilon_5 N_0^2$

in which ϵ_5 is bounded; and also that, by (18.3) we have

(19.8) $\quad dC'(\bar{\Delta} \mid B = B_0) = dC' - C''(B_0)\, dB.$

We are now ready to combine equations (19.2) through (19.8), inclusive. The process will be easier to follow if we perform it in several steps.

First, we transpose the second term of the left member of (19.2) to obtain

$$(19.9)\quad \Delta C(\bar{\Delta}|B = B_o) = \Delta C - \Delta B\cdot\bar{C}'(B_o + \Delta B) + \tfrac{1}{2}[\Delta B]^2\bar{C}''(B_1).$$

We then substitute (19.7) and (19.3) in (19.9). This yields

$$(19.10)\quad \Delta C(\bar{\Delta}|B=B_o) = \Delta C - \Delta B\ \bar{C}'(B_o) - [\Delta B]^2[\bar{C}''(B_o) - \tfrac{1}{2}C''(B_o)]$$
$$- \epsilon_5 N_o^2 \Delta B + \tfrac{1}{2}\epsilon_1 N_o^2 [\Delta B]^2.$$

But since neither $\bar{C}''(B_o) - C''(B_o)$ nor ΔB exceeds a bounded multiple of N_o, we conclude from (19.10) that

$$(19.11)\quad \Delta C(\bar{\Delta}|B=B_o) = \Delta C - \Delta B\cdot\bar{C}'(B_o) - \tfrac{1}{2}[\Delta B]^2\cdot C''(B_o) + \epsilon_6 N_o^3,$$

in which ϵ_6 is bounded.

Next, we substitute (19.8) in (19.4) and put the result in place of $\bar{C}'(B_o)$ in the right member of (19.11). We thus obtain

$$(19.12)\quad \Delta C(\bar{\Delta}|B=B_o) = \Delta C - \Delta B[C'(B_o) + dC' - C''(B_s)\ dB\ ]$$
$$- \tfrac{1}{2}[\Delta B]^2\ C''(B_o) + \epsilon_7 N_{1s}^3,$$

with ϵ_7 bounded, since $N_0 \leq N_{1s}$ and $\Delta B = O(N_0)$. But from (11.81) we have

$$(19.13) \qquad \Delta B = dB + \epsilon_8 N_{1s}^2$$

in which ϵ_8 is bounded; and, since from (9.2) it follows that

$$(19.14) \qquad dB \leq K_0 N_0,$$

we see from (19.13) and (19.14) that

$$(19.15) \qquad [\Delta B]^2 = (dB)^2 + \epsilon_9 N_{1s}^3,$$

with ϵ_9 bounded.

Now in the first two terms of the right member of (19.12) we replace ΔC, ΔB by the right members of (19.5), (19.6), respectively; and in the third term of the right member of (19.12) we replace $[\Delta B]^2$ by the right member of (19.15). We find in this way that

$$(19.16)\quad \Delta C(\bar{\Delta}|B=B_o) = dC + \frac{1}{2} d^2C + \epsilon_3 N_{2s}^3$$

$$- C'(B_o)\, dB - \frac{1}{2} C'(B_o)\, d^2B - C'(B_o)\, \epsilon_4 N_{2s}^3$$

$$- dC'dB - \frac{1}{2} dC'd^2B - dC'\ \epsilon_4 N_{2s}^3$$

$$+ C''(B_o)\, [dB]^2 + \frac{1}{2} C''(B_o)\, dBd^2B$$

$$+ C''(B_o)\, \epsilon_4 N_{2s}^3\, dB$$

$$- \frac{1}{2} C''(B_o)\, [dB]^2 - \frac{1}{2} \epsilon_9 N_{1s}^3\, C''(B_o) + \epsilon_7 N_{1s}^3.$$

But from (16.2) we see that d^2B does not exceed a bounded multiple of N_{1s}^2; so, since from (9.2) we have $dC' \leqq K_oN_o$, it follows from (19.16) that

$$(19.17)\quad \Delta C(\bar{\Delta}|B=B_o) = dC - C'(B_o)\, dB$$

$$+ \frac{1}{2}\left\{d^2C - C'(B_o)d^2B - 2\, dC'dB + C''(B_o)\,[dB]^2\right\}$$

$$+ \epsilon_{10} N_{2s}^3,$$

in which ϵ_{10} is bounded.

Finally, since $dC - C'(B_o)\, dB$ is by (13.3) the first differential $dC(\bar{\Delta}|\ B = B_o)$, it follows that the expression in braces $\{\cdots\}$ in the right member of (19.17) is $d^2C(\bar{\Delta}|\ B = B_o)$, the special form assumed by the second differential—defined in (12.8)—for the case $\Delta_1\Psi = \Delta_2\Psi = \bar{\Delta}$. This establishes (19.1).

As an illustration of the application of (19.1), let us find the second differential effect $d^2x(\overline{\Delta} \mid y = 0)$ on range due to the disturbance $\overline{\Delta}$. Just as in deducing the first differential range effect (18.4), we let $C = x$, $B = y$, $B_o = 0$, $A = t$, $A_o = T$. Hence $C'(B) = x'(y) = v_x(t)/v_y(t)$. Then, with the help of the normal differential equations of motion

$$\dot{v}_x = -Ev_x$$

$$\dot{v}_y = -Ev_y - g\ ,$$

we find that

$$(19.18) \quad C''(B_o) = x''(0) = \frac{v_y(T)\dot{v}_x(T) - v_x(T)\dot{v}_y(T)}{[v_y(T)]^3} = \frac{g\ v_x(T)}{[v_y(T)]^3}\ .$$

Moreover

$$(19.19) \quad dC'(\overline{\Delta} \mid A{=}A_o) = [v_y dv_x - v_x dv_y] / [v_y]^2\ ,$$

in which v_x, v_y are evaluated at $t = T$ and $dv_x = dv_x(\overline{\Delta} \mid t = T)$, $dv_y = dv_y(\overline{\Delta} \mid t = T)$. Substituting the appropriate values for C, B, . . . , dC' in (19.1) yields

$$(19.20)\quad d^2x(\bar{\Delta}|y = 0) = d^2x(\bar{\Delta}|t = T) - \frac{v_x(T)}{v_y(T)} d^2y(\bar{\Delta}|t = T)$$

$$- \frac{2\, dy(\bar{\Delta}|t = T)}{[v_y(T)]^2} \left\{ v_y(T)\, dv_x(\bar{\Delta}|t=T) - v_x(T)\, dv_y(\bar{\Delta}|t=T) \right\}$$

$$+ \frac{gv_x(T)}{[v_y(T)]^3} \left\{ dy(\bar{\Delta}|t = T) \right\}^2 \quad .$$

In case the striking velocity

$$v_\omega = \left\{ [v_x(T)]^2 + [v_y(T)]^2 \right\}^{1/2}$$

and the first differential effect $d\omega$ on the angle-of-fall ω have already been calculated for the trajectory to which one wishes to apply the formula (19.20), the latter equation can be written in a somewhat more conveniently applied form. For

$$(19.21)\qquad \omega = \pi - \theta(T)$$

in which $\theta(T)$ is the value at $t = T$ of the angle $\theta(t)$ of inclination of the tangent to the trajectory. Consequently, since

$$(19.22)\qquad C'(B) = x'(y) = v_x(t)/v_y(t) = \cot \theta(t),$$

we have $v_x(T)/v_y(T) = - \cot \omega$, and (19.18) may be written in the alternative form

$$(19.23) \quad C''(B_o) = x''(0)$$
$$= -g \cot \omega / [v_y(T)]^2$$
$$= -g \cot \omega \csc^2 \omega / v_\omega{}^2.$$

Moreover, from (19.22) we have

$$(19.24) \quad dC'(\bar{\Delta} | A = A_0) = - \csc^2 \theta(T) \, d\theta(\bar{\Delta} | t = T).$$

But the differential effect $d\omega$ on the angle-of-fall ω is the differential

$$(19.25) \quad d\omega = - d\theta(\bar{\Delta} | y = 0),$$

<u>not</u> $-d\theta(\bar{\Delta} | t = T)$. Fortunately, the relation (18.3) enables us to express this latter differential in terms of $d\omega$. However, to use (18.3) we first need to calculate the value of $d\theta/dy$ at $y = 0$ on the normal trajectory. To this end, we observe that

$$(19.26) \quad x''(y) = d(\cot\theta)/dy$$
$$= - \csc^2 \theta \, d\theta/dy,$$

from which it follows that

$$(19.27) \quad (d\theta/dy)\Big|_{y = 0} = - x''(0) \sin^2 \omega .$$

Now setting $C = \theta$, $B = y$, $B_o = 0$, $A = t$, $A_o = T$ in (18.3), and using (19.25), (19.27), we obtain

$$(19.28) \quad - d\omega = d\theta(\bar{\Delta} | t=T) + x''(0) \sin^2 \omega \, dy(\bar{\Delta} | t=T).$$

It follows from this last equation that (19.24) may be written as

$$(19.29) \quad dC'(\bar{\Delta} \mid A = A_0) = \csc^2\omega \, d\omega + x''(0) \, dy(\bar{\Delta} \mid t = T),$$

in which, we recall from (19.22), C' is $A'(y)$.

Consequently, when we set $C = x$, $B = y$, $B_0 = 0$, $A = t$, $A_0 = T$, and utilize (19.21), (19.22), (19.23), (19.29) in (19.1), we obtain

$$(19.30) \quad d^2x(\bar{\Delta} \mid y = 0) = d^2x(\bar{\Delta} \mid t = T) + \cot\omega \, d^2y(\bar{\Delta} \mid t = T)$$
$$+ (g/v_\omega^2) \cot\omega \csc^2\omega \left[dy(\bar{\Delta} \mid t=T)\right]^2$$
$$- 2 \csc^2\omega \, d\omega \, dy(\bar{\Delta} \mid t = T),$$

in which the differential $d\omega$ is that defined by (19.25). Thus (19.30) is an alternative form of (19.20).

Our improved estimate of the range effect ΔX caused by a disturbance $\bar{\Delta}$ is now

$$(19.31) \quad \Delta X \doteq dx(\bar{\Delta} \mid y = 0) + (1/2) \, d^2x(\bar{\Delta} \mid y = 0),$$

in which the first term of the right member is given by (18.4) and the second term is given either by (19.20) or by (19.30). Moreover, we have the assurance from (19.17) that the error of the approximation (19.31) is at most a bounded multiple of $[N_{2s}(\bar{\Delta})]^3$.

Chapter VI

REDUCTION OF THE SECOND VARIATIONS TO DOUBLE STIELTJES INTEGRALS

20. The Second Variation Equations and the Associated Homogeneous System.

If $q^i = q^i(s)$ is the standard trajectory defined by the normal equations (13.1), and if $\Delta\varphi$ is any set of disturbance functions, then in the conventional ballistic notation, let

$$(20.1) \quad e^i(s) \equiv F^i_{r\alpha}(\varphi(s, q(s)))\,\Delta\varphi^\alpha(s, q(s)),$$

so that the first variation equations (8.9) become, upon our writing Δ instead of Δ_1,

$$(20.2) \quad \begin{cases} \dfrac{d\eta^i}{ds} = F^i_{r\alpha}(\varphi(s,q(s)))\,\varphi^\alpha_{q^j}(s,q(s))\,\eta^j + e^i(s) \\ \eta^i(s_1) \equiv \Delta q^i_0, \quad (i = 1,\ldots,n\,;\ \alpha = 1,\ldots,m). \end{cases}$$

In order to apply to the <u>second</u> variations $\zeta^i(s) \equiv Q_{\tau\tau}(s,o)$ as much as possible of the existing theory concerning first variations, let us begin by using the symbol $\varepsilon^i(s)$ to denote the sum of the last five terms on the right member of the first of equations (16.1). That is,

$$(20.3)\quad \varepsilon^{i}(s) = \Big\{ F^{i}_{r^\alpha r^\beta}(\varphi(s,q(s)))\,\varphi^{\alpha}_{q^j}(s,q(s))\,\varphi^{\beta}_{q^k}(s,q(s))$$

$$+ F^{i}_{r^\alpha}(\varphi(s,q(s)))\,\varphi^{\alpha}_{q^j q^k}(s,q(s))\Big\}\,\eta^{j}(s)\,\eta^{k}(s)$$

$$+ 2F^{i}_{r^\alpha r^\beta}(\varphi(s,q(s)))\,\varphi^{\alpha}_{q^j}(s,q(s))\,\Delta\varphi^{\beta}(s,q(s))\,\eta^{j}(s)$$

$$+ F^{i}_{r^\alpha r^\beta}(\varphi(s,q(s)))\,\Delta\varphi^{\alpha}(s,q(s))\,\Delta\varphi^{\beta}(s,q(s))$$

$$+ 2F^{i}_{r^\alpha}(\varphi(s,q(s)))\,\Delta\varphi^{\alpha}_{q^j}(s,q(s))\,\eta^{j}(s).$$

Then (10.1) may be written as

$$(20.4)\quad \begin{cases} \dfrac{d\zeta^i}{ds} = F^{i}_{r^\alpha}(\varphi(s,q(s)))\,\varphi^{\alpha}_{q^j}(s,q(s))\,\zeta^{j} + \varepsilon^{i}(s) \\[2ex] \zeta^{i}(s_1) = 0. \end{cases}$$

It is at once apparent, from a comparison of (20.2) and (20.4), that these systems have the <u>same corresponding homogeneous system</u>

$$(20.5)\quad \frac{d\xi^i}{ds} = A_{ij}(s)\,\xi^{j},$$

in which

$$(20.6)\quad A_{ij}(s) = F^{i}_{r^\alpha}(\varphi(s,q(s)))\,\varphi^{\alpha}_{q^j}(s,q(s)),$$

as well as the same adjoint system

(20.7) $$\frac{d\lambda^i}{ds} = - A_{ji}(s)\,\lambda^j ,$$

the matrix $\| A_{ji}(s) \|$ being the transpose of the matrix $\| A_{ij}(s) \|$.

Now let $\xi^i_\nu(s)$, $\nu = 1, \ldots, n$, denote n linearly independent solutions of (20.5), and let $\Xi_\nu^{\,i}(s)$ denote the co-factor of $\xi_\nu^{\,i}(s)$ in the determinant

(20.8) $$D(s) = \begin{vmatrix} \xi^1_1(s) & \xi^1_2(s) & \cdots & \xi^1_n(s) \\ \xi^2_1(s) & \cdots & \cdots & \xi^2_n(s) \\ \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdot \\ \xi^n_1(s) & \cdots & \cdots & \xi^n_n(s) \end{vmatrix}$$

$$= D(s_1) \exp \int_{s_1}^{s} \left\{ A_{11}(s) + \ldots + A_{nn}(s) \right\} ds.$$

Then $D(s) \neq 0$, and it is well known that, for each ν, the functions

(20.9) $$\lambda^\nu_i(s) = \frac{\Xi_\nu^{\,i}(s)}{D(s)}, \quad (i = 1, \ldots, n)$$

constitute a solution of the adjoint system (20.7); and, moreover, that the functions $\eta^i(s)$ and $\zeta^i(s)$ may be expressed in terms of their respective initial values $\eta^i(s_1)$, $\zeta^i(s_1) = 0$, and the functions $\lambda^\nu_i(s)$ by means of the formulas

$$(20.10)\quad \eta^i(s) = \xi^i_\nu(s)\left\{ \lambda^\nu_r(s_1)\eta^r(s_1) + \int_{\sigma=s_1}^{\sigma=s} \lambda^\nu_r(\sigma)e^r(\sigma)\,d\sigma \right\}$$

and

$$(20.11)\quad \zeta^i(s) = \xi^i_\nu(s)\left\{ \lambda^\nu_p(s_1)\zeta^p(s_1) + \int_{\tau=s_1}^{\tau=s} \lambda^\nu_p(\tau)\,\varepsilon^p(\tau)\,d\tau \right\}$$

$$= \xi^i_\nu(s)\int_{\tau=s_1}^{\tau=s} \lambda^\nu_p(\tau)\;\varepsilon^p(\tau)\,d\tau.$$

In order to simplify the notation somewhat, we shall define functions $\kappa_\nu(s)$, $\mu_\nu(s)$ by writing

$$(20.12)\quad \kappa_\nu(s) = \lambda^\nu_r(s_1)\eta^r(s_1) + \int_{\sigma=s_1}^{\sigma=s} \lambda^\nu_r(\sigma)\;e^r(\sigma)\,d\sigma,$$

$$(r=1, \ldots, n)$$

and

$$(20.13)\quad \mu_\nu(s) = \int_{\tau=s_1}^{\tau=s} \lambda^\nu_p(\tau)\;\varepsilon^p(\tau)\,d\tau, \quad (p=1,\ldots,n).$$

We are then able to write (20.10) and (20.11) in the condensed forms

$$(20.14) \qquad \eta^i(s) = \xi^i_\nu(s)\,\kappa_\nu(s),$$

and

$$(20.15) \qquad \zeta^i(s) = \xi^i_\nu(s)\,\mu_\nu(s),$$

respectively. The functions $\kappa_\nu(s)$ are well known, and have been employed extensively in the calculation of (first) differential effects of abnormal forces and non-standard initial conditions. It is our purpose to examine, in the pages that follow, the analogous (but considerably more complicated) functions $\mu_\nu(s)$.

21. The Functions $\mu_\nu(s)$ as the Sum of Functions of Four Distinct Types.

We note first of all that the functions $\mu_\nu(s)$ are themselves dependent upon the functions $\kappa_\nu(s)$. In fact, when (20.14) is substituted in (20.3) and the result used in (20.13), we find that we can write

$$(21.1) \quad \mu_\nu(s) = \mu^{I}_\nu(s) + \mu^{II}_\nu(s) + \mu^{III}_\nu(s) + \mu^{IV}_\nu(s),$$

in which the terms of the right member are defined as follows:

$$(21.2)\ \mu_{\nu}^{I}(s) \equiv \int_{\tau=s_1}^{\tau=s} \lambda_p^{\nu}\left\{F^p_{r^\alpha r^\beta}\varphi^\alpha_{q^j}\varphi^\beta_{q^k} + F^p_{r^\alpha}\varphi^\alpha_{q^j q^k}\right\}\xi^j_\gamma \kappa_\gamma \xi^k_\delta \kappa_\delta \, d\tau,$$

$$(\gamma, \delta = 1, \ldots, n)$$

the functions λ^ν_p, ξ^j_γ, κ_γ, ξ^k_δ, κ_δ being evaluated for the argument τ, the functions $F^p_{r^\alpha}$ and $F^p_{r^\alpha r^\beta}$ for the argument $\varphi(\tau, q(\tau))$, and the functions $\varphi^\alpha_{q^j}$, $\varphi^\beta_{q^k}$, and $\varphi^\alpha_{q^j q^k}$ for the arguments τ, $q(\tau)$;

$$(21.3)\ \mu_{\nu}^{II}(s) \equiv 2\int_{\tau=s_1}^{\tau=s} \lambda^\nu_p F^p_{r^\alpha r^\beta}\varphi^\alpha_{q^j}\Delta\varphi^\beta \xi^j_\gamma \kappa_\gamma \, d\tau,$$

with $\Delta\varphi^\beta$ evaluated for the arguments τ, $q(\tau)$, and with the remaining functions in the integrand of (21.3) evaluated as described above for (21.2);

$$(21.4)\ \mu_{\nu}^{III}(s) \equiv \int_{\tau=s_1}^{\tau=s} \lambda^\nu_p F^p_{r^\alpha r^\beta}\Delta\varphi^\alpha\Delta\varphi^\beta \, d\tau,$$

with $\Delta\varphi^\alpha$ evaluated for the arguments τ, $q(\tau)$, and the other functions in the integrand of (21.4) evaluated in the manner already described; and, finally,

$$(21.5)\ \mu_{\nu}^{IV}(s) \equiv 2\int \lambda^\nu_p F^p_{r^\alpha}\Delta\varphi^\alpha_{q^j}\xi^j_\gamma \kappa_\gamma \, d\tau,$$

in which $\Delta\varphi^{\alpha}_{q^j}$ is evaluated for the arguments τ, $q(\tau)$, and the remaining functions are evaluated as previously described.

Thus our study of the functions $\mu_{\nu}(s)$ falls naturally into four parts; viz., an examination of each of the functions $\mu^{I}_{\nu}(s)$, $\mu^{II}_{\nu}(s)$, $\mu^{III}_{\nu}(s)$, $\mu^{IV}_{\nu}(s)$, separately. This we proceed to do.

22. The Functions $\mu^{I}_{\nu}(s)$.

In this section, we shall find it convenient to use the following abbreviated notation:

$$(22.1)\quad \psi^{\nu}_{jk}(\tau) = \lambda^{\nu}_{p}\left\{ F^{p}_{r^{\alpha} r^{\beta}}\, \varphi^{\alpha}_{q^j}\, \varphi^{\beta}_{q^k} + F^{p}_{r^{\alpha}}\, \varphi^{\alpha}_{q^j q^k} \right\},$$

the functions in the right member being evaluated as described above for (21.2). We can thus write (21.2) somewhat more succinctly as

$$(22.2)\quad \mu^{I}_{\nu}(s) = \int_{\tau=s_1}^{\tau=s} \psi^{\nu}_{jk}\, \xi^{j}_{\gamma}\, \kappa_{\gamma}\, \xi^{k}_{\delta}\, \kappa_{\delta}\, d\tau,$$

in which each of the functions appearing in the integrand is evaluated for the argument τ. If we now use (20.12) to substitute for $\kappa_{\gamma}(\tau)$ and $\kappa_{\delta}(\tau)$ in (22.2), the result is

$$(22.3)\quad \mu_\nu^{I}(s) = \int_{\tau=s_1}^{\tau=s} \psi_{jk}^{\nu}(\tau)\xi_\gamma^j(\tau)\,\xi_\delta^k(\tau)\,\kappa_\gamma(s_1)\,\kappa_\delta(s_1)\,d\tau$$

$$+\,2\int_{\tau=s_1}^{\tau=s} \psi_{jk}^{\nu}(\tau)\,\xi_\gamma^j(\tau)\,\xi_\delta^k(\tau)\,\kappa_\gamma(s_1)\left\{\int_{\sigma=s_1}^{\sigma=\tau}\lambda_u^\delta(\sigma)\,e^u(\sigma)\,d\sigma\right\}d\tau$$

$$+\int_{\tau=s_1}^{\tau=s} \psi_{jk}^{\nu}(\tau)\,\xi_\gamma^j(\tau)\,\xi_\delta^k(\tau)\left\{\int_{\sigma=s_1}^{\sigma=\tau}\lambda_r^\gamma(\sigma)\,e^r(\sigma)\,d\sigma\int_{\rho=s_1}^{\rho=\tau}\lambda_u^\delta(\rho)\,e^u(\rho)\,d\rho\right\}d\tau.$$

Of the three terms in the right member of (22.3), the first gives $\mu_\nu^I(s)$ when the disturbance consists <u>only</u> of <u>altered initial conditions</u>; the third furnishes $\mu_\nu^I(s)$ when the <u>initial conditions are standard</u> but are <u>accompanied by changes in the functions</u> $\varphi^\alpha(s, q)$, such as would be the case if <u>abnormal forces</u> were acting upon the projectile; while all three terms must be evaluated when one wishes the <u>combined effects</u> both of non-standard initial conditions and of abnormal forces. For later reference, we shall designate the three terms in the right member of (22.3) as $\mu_\nu^{I1}(s)$, $\mu_\nu^{I2}(s)$, and $\mu_\nu^{I3}(s)$, respectively. Since it is the third of these terms which is the most complicated, and since it is also the most important in applications, we shall turn our attention first to

$$(22.4)\ \mu_{\nu}^{I_3}(s) \equiv \int_{\tau=s_1}^{\tau=s} \psi^{\nu}_{jk}(\tau)\, \xi^{j}_{\gamma}(\tau)\, \xi^{k}_{\delta}(\tau) \left\{ \int_{\sigma=s_1}^{\sigma=\tau} \lambda^{\gamma}_{r}(\sigma) e^{r}(\sigma)\, d\sigma \int_{\rho=s_1}^{\rho=\tau} \lambda^{\delta}_{u}(\rho) e^{u}(\rho)\, d\rho \right\} d\tau .$$

Now, the right member of (22.4) is the sum of many integrals, each of which — by virtue of (20.1) — has the form

$$(22.5)\ I(s_1,s) \equiv \int_{\tau=s_1}^{\tau=s} \Psi(\tau) \left\{ \int_{\sigma=s_1}^{\sigma=\tau} \Phi(\sigma)\, \Delta\varphi^{\alpha}(\sigma, q(\sigma))\, d\sigma \int_{\rho=s_1}^{\rho=\tau} \Theta(\rho)\, \Delta\varphi^{\beta}(\rho, q(\rho))\, d\rho \right\} d\tau$$

In order to exhibit the functions $\Psi(\tau)$, $\Phi(\sigma)$, $\Theta(\rho)$ explicitly, as well as for the purpose of discussing the integral $I(s_1,s)$ and similar expressions which will occur in later sections, we shall find it convenient occasionally to abandon, temporarily, the summation convention concerning repeated indices. For the sake of clarity, we shall adopt the device indicated in the following

Remark Concerning Notation:

When it is necessary to repeat an index upon which summation is not intended, we shall inclose that index in parentheses. When an index is repeated but not inclosed

in parentheses, the usual summation convention will be implied.

Thus, for example, because the index α ranges over the set of integers 1, . . . , m , the symbol

$$F^{i}_{r\alpha}(\varphi(\sigma,\ q(\sigma)\ )\ )\ \Delta\varphi^{\alpha}\ (\sigma,\ q(\sigma)\ )$$

will mean the sum

$$F^{i}_{r1}(\varphi(\sigma,q(\sigma)))\Delta\varphi^{1}(\sigma,q(\sigma))+\cdots+F^{i}_{rm}(\varphi(\sigma,q(\sigma)))\Delta\varphi^{m}(\sigma,q(\sigma))$$

as heretofore; whereas, on the other hand, the symbol

$$F^{i}_{r(\alpha)}(\varphi(\sigma,q(\sigma)))\ \Delta\varphi^{(\alpha)}(\sigma,\ q(\sigma)\ )$$

will indicate any one term in the aforementioned sum.

It is our purpose to show that each term of the sum in the right member of (22.4) can be expressed as a double Stieltjes integral, and $I(s_1,s)$ is a typical such term. In the notation described on the preceding page, it follows from (22.4) and (20.1) that the functions $\Psi(\tau)$, $\Phi(\sigma)$, $\Theta(\rho)$ in (22.5) are

$$(22.6)\quad \begin{cases} \Psi(\tau) = \psi^{\nu}_{(j)(k)}(\tau)\,\xi^{(j)}_{(\gamma)}(\tau)\,\xi^{(k)}_{(\delta)}(\tau), \\ \Phi(\sigma) = \lambda^{(\gamma)}_{(i)}(\sigma)\,F^{(i)}_{r(\alpha)}(\varphi(\sigma, q(\sigma))), \\ \Theta(\rho) = \lambda^{(\delta)}_{(m)}(\rho)\,F^{(m)}_{r(\beta)}(\varphi(\rho, q(\rho))). \end{cases}$$

Our first step toward expressing $\mu_{\nu}^{I_3}(s)$ as a double Stieltjes integral will be to show that (22.5) can be written as

$$(22.7)\quad I(s_1,s) = \int_{\sigma=s_1}^{\sigma=s}\int_{\rho=s_1}^{\rho=s} \overline{\Psi}(s;\rho,\sigma)\,\Phi(\sigma)\,\Delta\varphi^{(\alpha)}(\sigma,q(\sigma))\,\Theta(\rho)\,\Delta\varphi^{(\beta)}(\rho,q(\rho))\,d\rho$$

in which

$$(22.8)\quad \overline{\Psi}(s;\rho,\sigma) \equiv \int_{\tau=\max(\rho,\sigma)}^{\tau=s} \Psi^{r}(\tau)\,d\tau \equiv \begin{cases} \displaystyle\int_{\tau=\sigma}^{\tau=s} \Psi^{r}(\tau)\,d\tau, & s_1 \leq \rho < \sigma; \\ \displaystyle\int_{\tau=\rho}^{\tau=s} \Psi^{r}(\tau)\,d\tau, & \sigma \leq \rho \leq s. \end{cases}$$

That is, we shall show that

$$(22.9)\quad I(s_1,s) = \int_{\sigma=s_1}^{\sigma=s}\int_{\rho=s_1}^{\rho=\sigma}\left\{\int_{\tau=\sigma}^{\tau=s}\Psi(\tau)\,d\tau\right\}\Phi(\sigma)\,\Delta\varphi^{(\alpha)}(\sigma,q(\sigma))\,\Theta(\rho)\,\Delta\varphi^{(\beta)}(\rho,q(\rho))\,d\rho\,d\sigma$$

$$+\int_{\sigma=s_1}^{\sigma=s}\int_{\rho=\sigma}^{\rho=s}\left\{\int_{\tau=\rho}^{\tau=s}\Psi(\tau)\,d\tau\right\}\Phi(\sigma)\,\Delta\varphi^{(\alpha)}(\sigma,q(\sigma))\,\Theta(\rho)\,\Delta\varphi^{(\beta)}(\rho,q(\rho))\,d\rho\,d\sigma.$$

That this is indeed true may perhaps most readily be seen by reference to Figure 1 on page 88. In fact, is follows at once from (22.5) that

$$(22.10)\quad I(s_1,s)=\int_{\tau=s_1}^{\tau=s}\int_{\sigma=s_1}^{\sigma=\tau}\int_{\rho=s_1}^{\rho=\tau}\Psi(\tau)\,\Phi(\sigma)\,\Delta\varphi^{(\alpha)}(\sigma,q(\sigma))\,\Theta(\rho)\,\Delta\varphi^{(\beta)}(\rho,q(\rho))\,d\rho\,d\sigma\,d\tau;$$

and it is clear from (22.10) that the three-dimensional region of integration is the "inverted" pyramid O-ABCD sketched in Figure 1, the bounding planes thereof being

OCD: $\rho = s_1$, OAB: $\rho = \tau$,

OAD: $\sigma = s_1$, OBC: $\sigma = \tau$,

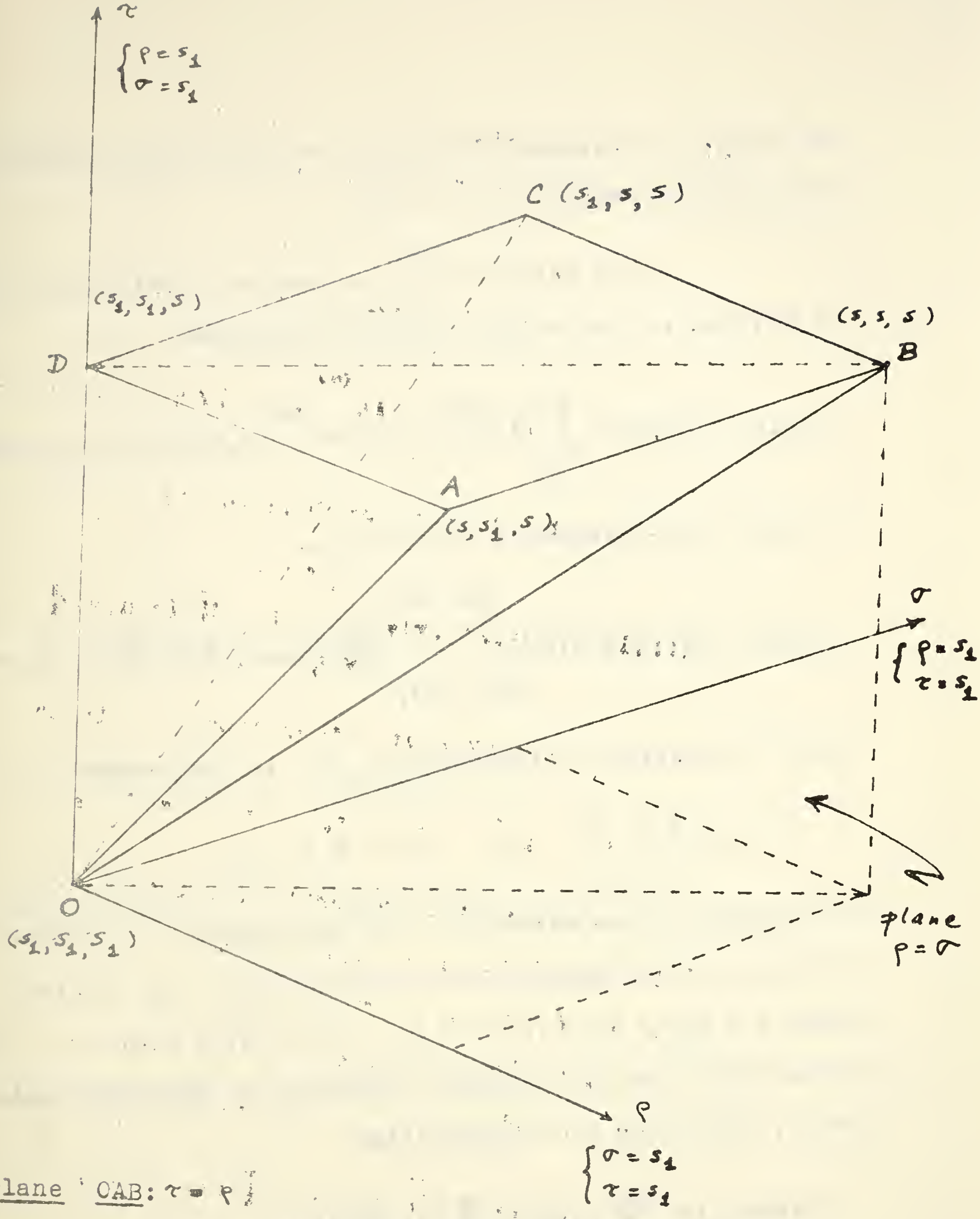
and

ABCD: $\tau = s_1$.

Now, a glance at the sketch reveals that this region is the sum of the two triangular pyramids O-ABD and O-BCD, which have face OBD ($\rho = \sigma$) in common. But the integrals of the function

$$\Psi(\tau)\,\Phi(\sigma)\,\Delta\varphi^{(\alpha)}(\sigma,\ q(\sigma))\,\Theta(\rho)\,\Delta\varphi^{(\beta)}(\rho,\ q(\rho))$$

over the regions O-BCD and O-ABD are, respectively, the first and second integrals occurring in the right member



Plane OAB: $\tau = \rho$

Plane OBC: $\tau = \sigma$

Plane OBD: $\rho = \sigma$

Figure 1.

of (22.9). Consequently (22.5) and (22.7) are equivalent, which is what we wished to show.

In the second place, we assert that (22.7) can be written as the double Stieltjes integral

$$(22.11)\quad I(s_1,s) = \iint_J \Delta\varphi^{(\alpha)}(\sigma, q(\sigma))\,\Delta\varphi^{(\beta)}(\rho, q(\rho))\,dg(s_1,s;\rho,\sigma),$$

in which the integrator function is

$$(22.12)\quad g(s_1,s;\rho,\sigma) = \int_{v=s_1}^{v=\sigma}\int_{u=s_1}^{u=\rho} \overline{\Psi}(s;u,v)\,\Phi(v)\,\Theta(u)\,du\,dv$$

and the interval of integration J is the square

$$J:\quad s_1 \leqq \rho \leqq s,\qquad s_1 \leqq \sigma \leqq s\,.$$

To establish this assertion, let the square J be subdivided into hf rectangular sub-intervals I_{ij} by the lines $\rho = \rho_i$, $\sigma = \sigma_j$, $(i = 1, \ldots, h;\ j = 1, \ldots, f)$. Then, employing the customary notation of Stieltjes integrals, and using the abbreviation

$$F(s;u,v) \equiv \overline{\Psi}(s;u,v)\,\Phi(v)\,\Theta(u),$$

we have

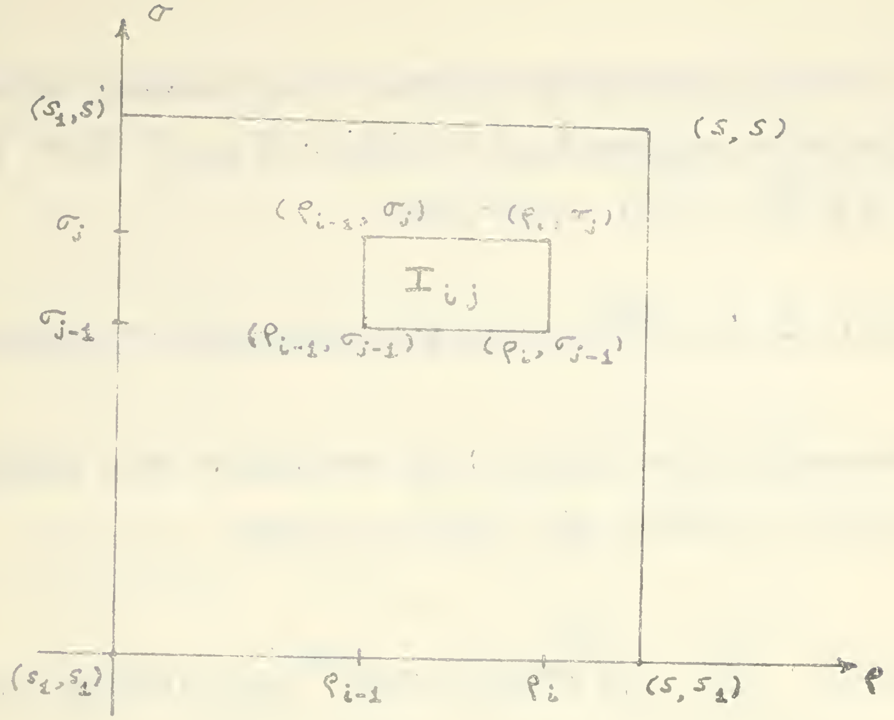


Figure 2.

$$(22.13)\quad \Delta_g I_{ij} = \int_{s_1}^{\sigma_j}\int_{s_1}^{\rho_i} F(s;u,v)\,du\,dv - \int_{s_1}^{\sigma_{j-1}}\int_{s_1}^{\rho_i} F(s,u,v)\,du\,dv$$

$$+\int_{s_1}^{\sigma_{j-1}}\int_{s_1}^{\rho_{i-1}} F(s;u,v)\,du\,dv - \int_{s_1}^{\sigma_j}\int_{s_1}^{\rho_{i-1}} F(s;u,v)\,du\,dv$$

$$= \int_{\sigma_{j-1}}^{\sigma_j}\int_{\rho_{i-1}}^{\rho_i} \overline{\Psi}(s;u,v)\,\Phi(v)\,\Theta(u)\,du\,dv.$$

But, by the mean value theorem of the integral calculus, there exist numbers ξ_i, η_j with $\rho_{i-1} \leq \xi_i \leq \rho_i$, $\sigma_{j-1} \leq \eta_j \leq \sigma_j$ such that

$$(22.14) \quad \Delta_g I_{ij} = \overline{\Psi}(s;\xi_i,\eta_j)\,\Phi(\eta_j)\,\Theta(\xi_i)\,[\sigma_j-\sigma_{j-1}][\rho_i-\rho_{i-1}].$$

Consequently from (22.14), the definition of a double Stieltjes integral, and (22.7), we have

$$(22.15) \quad \iint_{J} \Delta\varphi^{(\alpha)}(\sigma,q(\sigma))\,\Delta\varphi^{(\beta)}(\rho,q(\rho))\,dg(s_1,s;\rho,\sigma)$$

$$= \lim_{\substack{h\to\infty\\ f\to\infty}} \sum_{i=1}^{h}\sum_{j=1}^{f} \Delta\varphi^{(\alpha)}(\eta_j,q(\eta_j))\,\Delta\varphi^{(\beta)}(\xi_i,q(\xi_i))\,\Delta_g I_{ij}$$

$$= \lim_{\substack{h\to\infty\\ f\to\infty}} \sum_{i=1}^{h}\sum_{j=1}^{f} \overline{\Psi}(s;\xi_i,\eta_j)\,\Phi(\eta_j)\,\Delta\varphi^{(\alpha)}(\eta_j,q(\eta_j))\,\Theta(\xi_i)\,\Delta\varphi^{(\beta)}(\xi_i,q(\xi_i))\,[\sigma_j-\sigma_{j-1}][\rho_i-\rho_{i-1}]$$

$$= \int_{\sigma=s_1}^{\sigma=s}\int_{\rho=s_1}^{\rho=s} \overline{\Psi}(s;\rho,\sigma)\,\Phi(\sigma)\,\Delta\varphi^{(\alpha)}(\sigma,q(\sigma))\,\Theta(\rho)\,\Delta\varphi^{(\beta)}(\rho,q(\rho))\,d\rho\,d\sigma$$

$$= I(s_1, s),$$

which completes the proof of (22.11). Thus we have established the

(22.16) Theorem: Given ν, for each ordered pair of indices (α, β) on disturbance functions, and for each set $(j, k, \gamma, \delta, i, m)$ of the other 6 indices in (22.4), using the notation of (22.6), there is a corresponding term $I^{\nu}_{\alpha\beta jk\gamma\delta im}(s_1, s)$ in the right member of (22.4) which is expressible as a double Stieltjes integral of the form (22.11). Furthermore, all these integrals [corresponding to different sets $(j, k, \gamma, \delta, i, m)$ of the latter indices] have the same integrand, viz.,

$$\Delta\varphi^{\alpha}(\sigma, q(\sigma))\ \Delta\varphi^{\beta}(\rho, q(\rho)),$$

but each has its own integrator function depending upon the set $(j, k, \gamma, \delta, i, m)$ and given by (22.12), (22.8), and (22.6).

By a well known theorem on Stieltjes integrals, however, the sum of the integrals just referred to is equal to the integral of the function $\Delta\varphi^{\alpha}(\sigma, q(\sigma))\Delta\varphi^{\beta}(\rho, q(\rho))$ with respect to the sum of the separate integrator functions. That is, returning now to the summation convention regarding the repeated indices $j, k, \gamma, \delta, i, m, \vartheta, \psi$, if we define $g^{\nu I}_{\alpha\beta}(s_1, s; \rho, \sigma)$ by the equation

$$(22.17)\quad \mathfrak{S}^{\nu I}_{\alpha\beta}(s_1, s; \rho, \sigma) = \int_{v=s_1}^{v=\sigma} \int_{u=s_1}^{u=\rho} \overline{\Psi}^{\nu}_{\gamma\delta}(s; u, v)\, \Phi^{\gamma}_{\alpha}(v)\, \Theta^{\delta}_{\beta}(u)\, du\, dv,$$

wherein [Cf. (22.12), (22.8), (22.6), and (22.1)]

$$(22.18)\quad \overline{\Psi}^{\nu}_{\gamma\delta}(s; u, v) = \int_{\tau=\max(u,v)}^{\tau=s} \lambda^{\nu}_{p} \left\{ F^{p}_{r\vartheta r\psi}\, \varphi^{\vartheta}_{q^j}\, \varphi^{\psi}_{q^k} + F^{p}_{r\vartheta}\, \varphi^{\vartheta}_{q^j q^k} \right\} d\tau$$

[with the functions in the integrand evaluated as in (21.2)],

$$(22.19)\quad \Phi^{\gamma}_{\alpha}(v) = \lambda^{\gamma}_{i}(v)\, F^{i}_{r\alpha}(\varphi(v, q(v)))\quad ,$$

and

$$(22.20)\quad \Theta^{\delta}_{\beta}(u) = \lambda^{\delta}_{m}(u)\, F^{m}_{r\beta}(\varphi(u, q(u)))\quad ,$$

then, <u>for each ordered pair of indices</u> α, β <u>on disturbance functions the sum of the corresponding terms in the right member of</u> (22.4), <u>is</u>

$$(22.21)\quad \iint_{J} \Delta\varphi^{(\alpha)}(\sigma, q(\sigma))\Delta\varphi^{(\beta)}(\rho, q(\rho))\, d\mathfrak{g}^{\nu I}_{(\alpha)(\beta)}(s_1; s; \rho, \sigma).$$

Finally, since the right member of (22.4) consists of the sum of <u>all</u> such integrals as (22.21) for all possible ordered pairs α, β, [$\alpha, \beta = 1, \ldots, m$] , it follows that, extending the summation convention to the indices α, β, the equation (22.4) may be written as

$$(22.22)\quad \mu_\nu^{I4}(s) = \iint_{J} \Delta q^{\alpha}(\sigma, q(\sigma))\Delta q^{\beta}(\rho, q(\rho))\, dg^{\nu I}_{\alpha\beta}(s_1, s; \rho, \sigma).$$

in which the interval of integration J is the square

$$J : (s_1 \leqq \rho \leqq s; \qquad s_1 \leqq \sigma \leqq s)$$

it being understood that the right member of (22.22) is the sum of m^2 double Stieltjes integrals, one for each ordered pair (α, β) of the indices α and β.

Logically, perhaps, we should at this time discuss the other two terms in the right member of (22.3). But, because of their form, these terms are more easily handled after the functions $\mu_\nu^{II}(s)$, $\mu_\nu^{III}(s)$ have been treated. For the methods by which these latter functions are reduced to double Stieltjes integrals may be applied readily to the remaining terms in (22.3). This will be done in § 26.

23. The Functions $\mu_\nu^{II}(s)$.

Upon using (20.12) to substitute for $K_\nu(\tau)$ in (21.3), and then utilizing (20.1), we find that

$$(23.1)\quad \mu_\nu^{II}(s) \equiv 2\,\mu_\nu^{II1}(s) + 2\,\mu_\nu^{II2}(s)$$

in which

$$(23.2)\ \mu^{II_1}_{\nu}(s) \equiv \int_{\tau=s_1}^{\tau=s} K_{\gamma}(s_1)\, L^{\nu}_{\alpha\gamma}(\tau)\, \Delta\varphi^{\alpha}(\tau,\, q(\tau))\, d\tau,$$

and

$$(23.3)\ \mu^{II_2}_{\nu}(s) \equiv \int_{\tau=s_1}^{\tau=s} \int_{\sigma=s_1}^{\sigma=\tau} L^{\nu}_{\alpha\gamma}(\tau) M^{\gamma}_{\beta}(\sigma) \Delta\varphi^{\alpha}(\tau, q(\tau) \Delta\varphi^{\beta}(\sigma, q(\sigma)\ d\sigma d\tau,$$

wherein, in turn,

$$(23.4)\ L^{\nu}_{\alpha\gamma}(\tau) \equiv \lambda^{\nu}_{p}(\tau) F^{p}_{r\delta_r\alpha}(\varphi(\tau, q(\tau)))\varphi^{\delta}_{q^j}(\tau, q(\tau))\ \xi^{j}_{\gamma}(\tau),$$

and [Cf. (22.19), (22.30)]

$$(23.5)\quad M^{\gamma}_{\beta}(\sigma) \equiv \lambda^{\gamma}_{i}(\sigma)\ F^{i}_{r\beta}\ (\varphi(\sigma,\ q(\sigma)\ )\ ).$$

Now the integrals $\mu^{II_1}_{\nu}(s)$ are formally like those designated $\mu^{III}_{\nu}(s)$ in (21.4), in that only a single quadrature is involved; consequently we shall defer treating them until after the functions $\mu^{III}_{\nu}(s)$ have been discussed.

We turn our attention, then, to (23.3) and note that, as in the case of (22.4), its right member is the sum of a multitude of integrals. Each such term in this sum, however, has the form

$$(23.6)\ J\ (s_1, s) \equiv \int_{\tau=s_1}^{\tau=s} \int_{\sigma=s_1}^{\sigma=\tau} L(\tau) M(\sigma) \Delta\varphi^{\alpha}(\tau, q(\tau)) \Delta\varphi^{\beta}(\sigma, q(\sigma)) d\sigma d\tau.$$

In fact, given ν, for each ordered pair (α, β) of the indices on disturbance functions, and for every set (p, δ, j, γ, i) of values of the other 5 indices, the corresponding functions $L(\tau)$ and $M(\sigma)$ are given by (23.4) and (23.5), respectively. Our next task is to show that the typical integral $J(s_1, s)$ defined in (23.6) can be written as the double Stieltjes integral

$$(23.7) \quad \iint_S \Delta\varphi^{\alpha}(\tau, q(\tau))\Delta\varphi^{\beta}(\sigma, q(\sigma)) \, dg\,(s_1; \sigma, \tau),$$

in which the integrator function $g(s_1; \sigma, \tau)$ is

$$(23.8) \quad g(s_1; \sigma, \tau) = \int_{v=s_1}^{v=\tau} \int_{u=s_1}^{u=\sigma} L(v)\, M(u)\, K_E(u, v)\, du\, dv,$$

the function $K_E(\sigma, \tau)$, defined by

$$(23.9) \quad K_E(\sigma, \tau) = \begin{cases} 0, & s_1 \leq \tau < \sigma\ ; \\ 1, & \sigma \leq \tau \leq s\ , \end{cases}$$

being the <u>characteristic</u> <u>function</u> of the set

$$E = \{(\sigma, \tau) \mid s_1 \leq \sigma \leq \tau \leq s\}$$

which is the closed triangle OAB in the figure on page 97.

To this end, let ϵ be any positive number. Let N_0 denote the norm $N_0(\Delta q_0, \Delta\varphi)$ and let $\delta\eta$ be an

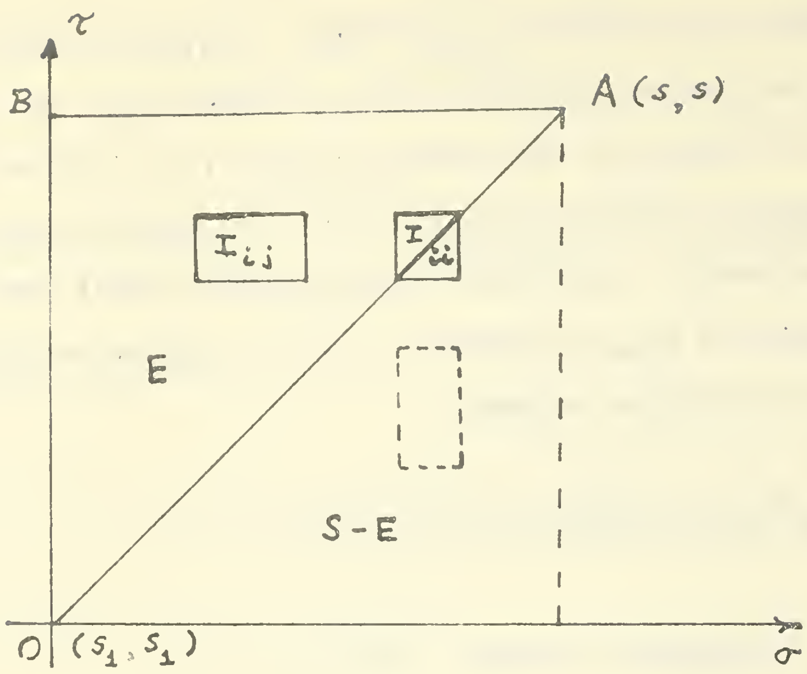


Figure 3.

upper bound for the absolute values of the product $L(v)M(u)K_E(u,v)$ on the square S: $(s_1 \leqq \sigma \leqq s,\ s_1 \leqq \tau \leqq s)$. By first choosing an integer h such that

$$h > 2\eta N_0^2(s - s_1)^2 / \epsilon ,$$

we next partition S into h^2 squares I_{ij} by means of the lines

$$\sigma = \sigma_i, \quad \tau = \tau_j,$$

wherein

$$\sigma_i = s_1 + i\left(\frac{s - s_1}{h}\right)$$

$$\tau_j = s_1 + j\left(\frac{s - s_1}{h}\right)$$

for $i = 1, \ldots, h$, and $j = 1, \ldots, h$. Then the sum of the areas of the h squares I_{ii} $(i = 1, \ldots, h)$ is evidently

$$\frac{(s - s_1)^2}{h} < \frac{\epsilon}{2\eta N_0^2}$$

Now the double Stieltjes integral (23.7) is the limit, as $h \to \infty$, of the double sum

$$(23.10) \quad \sum_{i=1}^{h} \sum_{j=1}^{h} \Delta\varphi^{\alpha}(\xi_j, q(\xi_j))\, \Delta\varphi^{\beta}(\xi_i, q(\xi_i))\, \Delta_g I_{ij},$$

in which (ξ_i, ξ_j) is any point—to be specified more carefully later—of the square I_{ij}. But this double sum may clearly be regarded as the sum of two parts Σ_1, Σ_2 defined as follows: Σ_1 consists of those terms for which $i = j$; Σ_2 consists of all the remaining terms. It is easy to see that $\Sigma_1 < \epsilon$. For from (23.8) we have

$$(23.11) \quad \Delta_g I_{ij} = g(s_1;\sigma_i,\tau_j) - g(s_1;\sigma_i,\tau_{j-1}) + g(s_1;\sigma_{i-1},\tau_{j-1}) - g(s_1;\sigma_{i-1},\tau_j)$$

$$= \int_{v=\tau_{j-1}}^{v=\tau_j} \int_{u=\sigma_{i-1}}^{u=\sigma_i} L(v)E(u)K_E(u,v)\,du\,dv;$$

and, by a well known integral estimate*, the iterated integral in the right member of (23.11) is not greater than the product of $\mathfrak{M}$ by the area of the square I_{ij}. Since the terms in Σ_1 are those which correspond to the squares I_{ii}, we need only compare each such term with the product of $N_0^2 \mathfrak{M}$ by the area of the corresponding square I_{ii} to show that Σ_1 is not greater than the product of $N_0^2 \mathfrak{M}$ by the sum of the areas of the squares I_{ii}. But since this latter sum is less than $\epsilon/2\mathfrak{M} N_0^2$, it follows that

$$(23.12) \qquad \Sigma_1 < \epsilon .$$

Now let $\epsilon \to 0$. Then from the definition of h it follows that $h \to \infty$. Moreover from (23.12) we have

$$\lim_{\epsilon \to 0} \Sigma_1 = 0.$$

But then, since the double Stieltjes integral in (23.7) is the limit, as $h \to \infty$, of the double sum (23.10), and

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* Courant: Differential and Integral Calculus, Vol. II, p. 232.

since this double sum is $\Sigma_1 + \Sigma_2$, it follows that the double Stieltjes integral in (23.7) is equal to

$$\lim_{h \to \infty} \Sigma_2 .$$

The terms in Σ_2 being those terms in (23.10) for which $i \neq j$, we conclude from (23.9), (23.10), and (23.11) that all the terms in Σ_2 vanish with the exception of those which correspond to squares I_{ij} lying entirely within the triangle E pictured in Figure 3. On E the integrand $L(v)M(u)K_E(u,v)$ is continuous; so the theorem of the mean assures us the existence of a pair of numbers ξ_i, η_j such that $\sigma_{i-1} \leqq \xi_i \leqq \sigma_i$, $\tau_{j-1} \leqq \eta_j \leqq \tau_j$, and for which the equation

$$(23.13) \quad \Delta_{\beta} I_{ij} = L(\eta_j) M(\xi_i) K_E(\xi_i, \eta_j) [\sigma_i - \sigma_{i-1}] \; [\tau_j - \tau_{j-1}]$$

is valid when I_{ij} lies entirely within the region E. Consequently the double Stieltjes integral in (23.7) is equal to

$$(23.14) \quad \lim_{h \to \infty} \sum_{\substack{i=1 \\ (i \neq j)}}^{h} \sum_{j=1}^{h} \Delta\varphi^{\alpha} \Delta\varphi^{\beta} L\, M\, K_E [\sigma_i - \sigma_{i-1}] \; [\tau_j - \tau_{j-1}] ,$$

with $\Delta\varphi^{\alpha}$ evaluated for the arguments $(\eta_j,\ q(\eta_j))$, $\Delta\varphi^{\beta}$ for $(\xi_i,\ q(\xi_i))$, L for (η_j), M for (ξ_i), and K_E for (ξ_i, η_j). But the limit (23.14) is the iterated integral

in (23.6), which is therefore identical with the Stieltjes integral (23.7), as was to be proved.

We are now ready to combine all the terms in the right member of (23.3), by an argument quite analogous to that which was employed in deducing (22.22). We begin by defining, for a given ν, and a given ordered pair (α, β) of indices on disturbance functions, the function

$$(23.15)\quad g^{\nu II}_{\alpha\beta}(s_1;\sigma,\tau) = \int_{v=s_1}^{v=\tau}\int_{u=s_1}^{u=\sigma} L^{\nu}_{\alpha\gamma}(v)M^{\gamma}_{\beta}(u)K_E(u,v)\,du\,dv,$$

in which $L^{\nu}_{\alpha\gamma}(v)$, $M^{\gamma}_{\beta}(u)$, and $K_E(u,v)$ are obtained from (23.4), (23.5), and (23.9), respectively, and in which we again return to the summation convention for the repeated indices p, δ, j, γ, and i. Evidently (23.15) is the sum of all those integrator functions of the type (23.8) determined by the aggregate of indices p, δ, j, γ, i. By the theorem on addition of integrator functions for Stieltjes integrals, referred to in the remark following (22.16), it follows that, for each ν and each ordered pair of values of α and β, the sum of the corresponding terms in the right member of (23.3) is the integral

$$(23.16)\quad \iint_S \Delta\varphi^{(\alpha)}(\tau,q(\tau))\,\Delta\varphi^{(\beta)}(\sigma,q(\sigma))\,dg^{\nu II}_{(\alpha)(\beta)}(s_1;\sigma,\tau).$$

But the right member of (23.3) is the sum of all such integrals as (23.16), corresponding to all possible ordered

pairs (α, β) for $\alpha = 1, \ldots, m$, $\beta = 1, \ldots, m$. Hence, again extending the summation convention to the repeated indices α, β, we may now write (23.3) as

$$(23.17) \quad \mu_\nu^{II2}(s) = \iint_S \Delta\varphi^\alpha(\tau, q(\tau))\,\Delta\varphi^\beta(\sigma, q(\sigma))\,dg_{\alpha\beta}^{\nu II}(s1; \sigma, \tau).$$

In this last equation, the region S is the square $(s1 \leq \sigma \leq s;\ s1 \leq \tau \leq s)$ and the integrator function $g_{\alpha\beta}^{\nu II}(s1; \sigma, \tau)$ is given by (23.15). As in (22.22), the right member of (23.17) is the sum of m^2 integrals, one for each ordered pair (α, β).

The way is now open, of course, to combine those terms occurring in (22.22) and (23.17) which correspond to the same ordered pair (α, β) of indices on disturbance functions. But this step we shall postpone until after our discussion of the functions $\mu_\nu^{III}(s)$ in order that we may then combine all terms occurring in the right member of (21.1) which correspond to the same pair (α, β).

24. The Functions $\mu_\nu^{III}(s)$.

It would appear from (21.4) that the functions $\mu_\nu^{III}(s)$ are considerably simpler than either $\mu_\nu^{II2}(s)$ or $\mu_\nu^{I3}(s)$, in that only a single quadrature each is necessary for their evaluation. In order, however, to combine those terms in (21.4) determined by a particular pair (α, β), of indices on disturbance functions with

other similarly determined terms from (21.2) and (21.3), it will be desirable to have the functions $\mu_{\nu}^{III}(s)$ expressed also as double Stieltjes integrals over the same interval of integration as that employed for (21.2) and (21.3). This can be done, though somewhat greater care is necessary in the process.

In fact, if we let

$$(24.1)\qquad N_{\alpha\beta}^{\nu}(\tau) \equiv \lambda_p^{\nu}(\tau)\, F_{r^{\alpha} r^{\beta}}^{p}(\varphi(\tau,\, q(\tau)\,)\,),$$

then (21.4) can be written as

$$(24.2)\qquad \mu_{\nu}^{III}(s) \equiv \int_{\tau=s_1}^{\tau=s} N_{\alpha\beta}^{\nu}(\tau)\Delta\varphi^{\alpha}(\tau,q(\tau))\Delta\varphi^{\beta}(\tau,q(\tau)\,)\,d\tau.$$

But, given ν, to each ordered pair (α,β) of indices on disturbance functions there corresponds in the right member of (24.2) a term of the form

$$(24.3)\qquad K(s_1,s) \equiv \int_{\tau=s_1}^{\tau=s} N_{(\alpha)(\beta)}^{\nu}(\tau)\Delta\varphi^{(\alpha)}(\tau,q(\tau))\Delta\varphi^{(\beta)}(\tau,q(\tau))\,d\tau.$$

Therefore our next objective will be to show that $K(s_1,s)$ is expressible as the double Stieltjes integral

$$(24.4)\qquad \iint_{S} \Delta\varphi^{(\alpha)}(\tau,q(\tau))\Delta\varphi^{(\beta)}(\sigma,q(\sigma))\, dg_{(\alpha)(\beta)}^{\nu III}(s_1;\sigma,\tau),$$

in which the integrator function is

$$(24.5) \qquad g_{\alpha\beta}^{\nu\ III}(s_1;\sigma,\tau) = \int_{v=s_1}^{v=\min(\sigma,\tau)} N_{\alpha\beta}^{\nu}(v)\,dv.$$

To this end, we observe first that for each $\epsilon > 0$ we can find a partition of the square S such that, for every re-partition, the corresponding finite sum is within ϵ of the integral (24.4). Let $\alpha_0 < \alpha_1 < \dots < \alpha_h$ be all the numbers which occur either as σ- or as τ-coordinates of vertices of sub-intervals in a partition with this property. Consider the intervals

$$(24.6) \qquad I_{ij}: \left\{ (\sigma,\tau) \mid \alpha_{i-1} \leq \sigma \leq \alpha_i;\ \alpha_{j-1} \leq \tau \leq \alpha_j \right\},$$

for i = 1, . . . , h, and j = 1, . . . , h. For each one of these intervals, either <u>at most one</u> of its vertices is on the diagonal $\tau - s_1 = \sigma - s_1$, or else its diagonal lies on the diagonal of the square S. In fact, if $i \neq j$, the diagonal $\tau - s_1 = \sigma - s_1$ has at most one point in I_{ij}; whereas, for each i, I_{ii} is a square with its diagonal on the diagonal of S.

Now, in order to simplify the notation throughout the remainder of this section, we shall employ the abbreviations

$$(24.7)\quad \begin{cases} g(s_1;\sigma,\tau) = g^{\nu III}_{(\alpha)(\beta)}(s_1;\sigma,\tau), \\ N(\tau) = N^{\nu}_{(\alpha)(\beta)}(\tau). \end{cases}$$

Recalling that, by definition,

$$(24.8)\quad \iint_{S} \Delta\varphi^{(\alpha)}(\tau,q(\tau))\Delta\varphi^{(\beta)}(\sigma,q(\sigma))\, dg(s_1;\sigma,\tau)$$

$$= \lim_{h\to\infty}\sum_{i=1}^{h}\sum_{j=1}^{h}\Delta\varphi^{(\alpha)}(\eta_j,q(\eta_j))\Delta\varphi^{(\beta)}(\xi_i,q(\xi_i))\Delta_g I_{ij},$$

wherein (ξ_i,η_j) is any point of the interval I_{ij}, and that

$$(24.9)\quad \Delta_g I_{ij} = g(s_1;\sigma_i,\tau_j) - g(s_1;\sigma_i,\tau_{j-1})$$

$$+ g(s_1;\sigma_{i-1},\tau_{j-1}) - g(s_1;\sigma_{i-1},\tau_j),$$

it is easy to see that, using (24.5), the right member of (24.9) is identically zero for every interval I_{ij} for which $i \neq j$. For if I_{ij} belongs to the triangle $S_1:\{(\sigma,\tau) \mid s_1 \leqq \sigma \leqq \tau \leqq s\}$, then the smaller coordinate at each of its vertices is the corresponding σ, so that the terms in the right member of (24.9) cancel out by pairs. Similarly, if I_{ij} belongs to the triangle $S_2:\{(\sigma,\tau) \mid s_1 \leqq \tau \leqq \sigma \leqq s\}$, then the smaller coordinate at each of its vertices is the corresponding τ, and again the terms on the right in (24.9) cancel in pairs.

Hence we may confine our attention only to those terms of the double sum in (24.8) for which $i = j$.

But from (24.5), (24.7), and (24.9) we have

$$(24.10)\quad \Delta_g I_{ii} = \int_{s_1}^{\alpha_i} N(v)\,dv - \int_{s_1}^{\alpha_{i-1}} N(v)\,dv$$

$$+ \int_{s_1}^{\alpha_{i-1}} N(v)\,dv - \int_{s_1}^{\alpha_{i-1}} N(v)\,dv$$

$$= \int_{\alpha_{i-1}}^{\alpha_i} N(v)\,dv.$$

Moreover, since $N(v)$ is continuous, the mean value theorem of the integral calculus guarantees the existence of a number β_i with $\alpha_{i-1} \leqq \beta_i \leqq \alpha_i$ and such that

$$(24.11)\quad \int_{\alpha_{i-1}}^{\alpha_i} N(v)\,dv = N(\beta_i)\,[\alpha_i - \alpha_{i-1}]\,.$$

If we now choose both ξ_i and η_i to be β_i, it follows from (24.8), (24.10), and (24.11), and from our conclusions concerning those terms of the right member of (24.8) for which $i \neq j$, that

$$(24.12)\quad \iint_S \Delta\varphi^{(\alpha)}(\tau, q(\tau))\,\Delta\varphi^{(\beta)}(\sigma, q(\sigma))\,dg\,(s_1;\sigma,\tau)$$

$$= \lim_{h\to\infty} \sum_{i=1}^{h} \Delta\varphi^{(\alpha)}(\beta_i, q(\beta_i))\,\Delta\varphi^{(\beta)}(\beta_i, q(\beta_i))\,N(\beta_i)\,[\alpha_i - \alpha_{i-1}].$$

But the left member of (24.12) is the double Stieltjes integral (24.4); while the right member of (24.12), because $\alpha_{i-1} \leqq \beta_i \leqq \alpha_i$ and $s_1 = \alpha_0 < \ldots < \alpha_h = s$, is the integral $\mathcal{K}(s_1,s)$ of (24.3). Thus (24.12) establishes the conclusion we set out to prove concerning that integral in the right member of (24.12) which corresponds to a particular ordered pair of indices α, β on disturbance functions and a given index ν. If we extend the summation convention to the repeated indices α, β in (24.4), we obtain [Cf. (24.2)]

$$(24.13)\quad \mu_{\nu}^{III}(s) = \iint_S \Delta\varphi^{\alpha}(\tau, q(\tau))\Delta\varphi^{\beta}(\sigma, q(\sigma))\, dg_{\alpha\beta}^{\nu III}(s_1;\sigma,\tau),$$

in which the right member is the sum of m^2 integrals like (24.4), one for each ordered pair (α, β), $(\alpha = 1, \ldots, m;\ \beta = 1, \ldots, m)$.

25. The Functions $\mu_{\nu}^{IV}(s)$.

From (21.5), (20.12) and (20.1) we have

$$(25.1)\quad \mu_{\nu}^{IV}(s) \equiv 2\mu_{\nu}^{IV1}(s) + 2\mu_{\nu}^{IV2}(s),$$

in which

$$(25.2)\quad \mu_{\nu}^{IV1}(s) \equiv \int_{\tau=s_1}^{\tau=s} K_{\gamma}(s_1) P_{\alpha j \gamma}^{\nu}(\tau)\, \Delta\varphi_{q^j}^{\alpha}(\tau, q(\tau))\, d\tau,$$

and

$$(25.3)\ \mu^{IV_2}_{\nu}(s) \equiv \int_{\tau=s_1}^{\tau=s}\int_{\sigma=s_1}^{\sigma=\tau} P^{\nu}_{\alpha j \gamma}(\tau)\, M^{\gamma}_{\beta}(\sigma)\, \Delta\varphi^{\alpha}_{q^j}(\tau, q(\tau))\, \Delta\varphi^{\beta}(\sigma, q(\sigma))\, d\sigma\, d\tau,$$

wherein, in turn

$$(25.4)\quad P^{\nu}_{\alpha j \gamma}(\tau) \equiv \lambda^{\nu}_{p}(\tau)\, F^{p}_{r\alpha}(\tau, q(\tau))\, \xi^{j}_{\gamma}(\tau)$$

and [Cf. (23.5)]

$$(25.5)\quad M^{\gamma}_{\beta}(\sigma) \equiv \lambda^{\gamma}_{i}(\sigma)\, F^{i}_{r\beta}(\sigma, q(\sigma)).$$

If we now compare (25.3) with (23.3), we see that each integral in the right member of the former equation has the same form as do those in the latter; namely, that exhibited in (23.6)—provided, of course, that we replace $\Delta\varphi^{\alpha}(\tau, q(\tau))$ in (23.6) by $\Delta\varphi^{\alpha}_{q^j}(\tau, q(\tau))$ for each j (j = 1, . . . , n). But then the argument employed in establishing (23.16) and (23.17) leads us to conclude that, given ν, for each ordered pair (α, β) of indices on disturbance functions, and for each j, the sum of the corresponding terms in the right member of (25.3) is the double Stieltjes integral

$$(25.6)\ \iint_S \Delta\varphi^{(\alpha)}_{q^{(j)}}(\tau, q(\tau))\,\Delta\varphi^{(\beta)}(\sigma, q(\sigma))\,dg^{\nu\,IV}_{(\alpha)(\beta)(j)}(s1;\sigma,\tau),$$

wherein the integrator function $g^{\nu\,IV}_{\alpha\beta j}(s1;\sigma,\tau)$ is defined [Cf. (23.5)] as

$$(25.7)\ g^{\nu\,IV}_{\alpha\beta j}(s1;\sigma,\tau) = \int_{v=s_1}^{v=\tau}\int_{u=s_1}^{u=\sigma} P^{\nu}_{\alpha j\gamma}(v)\,H^{\gamma}_{\beta}(u)\,K_E(u,v)\,du\,dv\,.$$

The functions in the integrand of (25.7) are those defined in (25.4), (25.5), and (23.9). To obtain the sum of those terms in the right member of (25.3) which correspond to a given ν and an ordered pair of indices α, β on disturbance functions, we sum on the index j in the expression (25.6), thus obtaining, say,

$$(25.8)\ \mu^{\alpha\beta\,IV_1}_{\nu}(s) = \iint_S \Delta\varphi^{(\alpha)}_{q^j}(\tau, q(\tau))\,\Delta\varphi^{(\beta)}(\sigma, q(\sigma))\,dg^{\nu\,IV}_{(\alpha)(\beta)j}(s_1;\sigma,\tau).$$

Unfortunately it does not appear possible, in general, to combine these n integrals in (25.8) into a single term, at least not by the device of adding integrator functions. For the functions $\Delta\varphi^{\alpha}_{qj}(\tau, q(\tau))$ are conceivably different for different values of the index j. Extending the summation convention to the indices

α, β, however, it follows from (25.8) that we can write (25.3) as

$$(25.9)\quad \mu_{\nu}^{IV2}(s) = \iint_S \Delta\varphi^{\alpha}_{q^j}(\tau, q(\tau))\Delta\varphi^{\beta}(\sigma, q(\sigma))\, dg^{\nu\ IV}_{\alpha\beta j}(s_1; \sigma, \tau),$$

in which the right member is the sum of nm^2 double Stieltjes integrals, n such integrals for each ordered pair of indices α, β on disturbance functions.

On the other hand, for trajectories on which altitude y can be used as independent variable, and if the disturbance $\Delta\varphi$ depends only on altitude, then $\Delta\varphi_{q^j} \equiv 0$ and we do not have to contend with the functions (25.1).

The functions $\mu_{\nu}^{IV1}(s)$ defined in (25.2) involve a single quadrature for each term of the right member of that equation, and so are formally like the functions $\mu_{\nu}^{III}(s)$ which have already been discussed. They will be treated in the next section.

26. The Functions $\mu_{\nu}^{I1}(s)$, $\mu_{\nu}^{I2}(s)$, $\mu_{\nu}^{II1}(s)$, and $\mu_{\nu}^{IV1}(s)$.

In this section we shall attempt to show how the four types of functions, $\mu_{\nu}^{I1}(s)$, $\mu_{\nu}^{I2}(s)$, $\mu_{\nu}^{II1}(s)$ and $\mu_{\nu}^{IV1}(s)$ can be reduced to double Stieltjes integrals by the same methods previously developed for the functions $\mu_{\nu}^{II2}(s)$ and $\mu_{\nu}^{III}(s)$.

We shall consider first the functions [Cf. the first term of the right member of (22.3)]

$$(26.1)\quad \mu_{\nu}^{I1}(s) = \int_{\tau=s_1}^{\tau=s} \psi_{\cdot jk}^{\nu}(\tau)\,\xi_{\gamma}^{j}(\tau)\,\xi_{\delta}^{k}(\tau)\,\kappa_{\gamma}(s_1)\,\kappa_{\delta}(s_1)\,d\tau,$$

in which the abbreviation $\psi_{\cdot jk}^{\nu}(\tau)$ is defined in (22.1). Let

$$(26.2)\quad Q_{\gamma\delta}^{\nu}(\tau) = \psi_{\cdot jk}^{\nu}(\tau)\,\xi_{\gamma}^{j}(\tau)\,\xi_{\delta}^{k}(\tau),$$

so that (26.1) can be written as

$$(26.3)\quad \mu_{\nu}^{I1}(s) = \int_{\tau=s_1}^{\tau=s} \kappa_{\gamma}(s_1)\,\kappa_{\delta}(s_1)\,Q_{\gamma\delta}^{\nu}(\tau)\,d\tau.$$

Then, if we define $g_{\gamma\delta}^{\nu I1}(s_1;\sigma,\tau)$ by [Cf. (24.5)]

$$(26.4)\quad g_{\gamma\delta}^{\nu I1}(s_1;\sigma,\tau) = \int_{v=s_1}^{v=\min(\sigma,\tau)} Q_{\gamma\delta}^{\nu}(\tau)\,d\tau,$$

an argument similar to that which established the equivalence of (24.3) and (24.4) yields the conclusion that, for each ordered pair of values for the indices γ, δ, the sum of the corresponding terms in the right member of (26.1) is

$$(26.5)\quad \kappa_{(\gamma)}(s_1)\,\kappa_{(\delta)}(s_1) \iint_{S} dg_{(\gamma)(\delta)}^{\nu I1}(s_1;\sigma,\tau).$$

When we extend the summation convention to γ, δ, it follows from (26.5) that (26.3) can be written as

$$(26.6)\quad \mu_{\nu}^{I1}(s) = \kappa_{\gamma}(s1)\kappa_{\delta}(s1)\iint_{S} dg_{\gamma\delta}^{J2}(s1;\sigma,\tau), \quad [\gamma,\delta=1,\dots,n].$$

Secondly, we observe that the second term of the right member of (22.3) can be written as

$$(26.7)\quad \mu_{\nu}^{I2}(s) = 2\int_{\tau=s_1}^{\tau=s}\int_{\sigma=s_1}^{\sigma=\tau} \Psi_{jk}^{\nu}(\sigma)\xi_{\gamma}^{j}(\tau)\xi_{\delta}^{k}(\tau)\kappa_{\gamma}(s1)\lambda_{u}^{\delta}(\sigma)\epsilon^{u}(\sigma)\,d\sigma d\tau.$$

Now let

$$(26.8)\quad \Psi_{\gamma\delta}^{\nu}(\tau) = \Psi_{jk}^{\nu}(\tau)\,\xi_{\gamma}^{j}(\tau)\,\xi_{\delta}^{k}(\tau),$$

and [recalling (23.5)]

$$(26.9)\quad M_{\beta}^{\delta}(\sigma) = \lambda_{u}^{\delta}(\sigma)\,F_{\gamma\beta}^{u}(\varphi(\sigma,\,q(\sigma))).$$

Then (26.7) can be written, using (20.1), (26.8), and (26.9), in the form

$$(26.10)\quad \mu_{\nu}^{I2}(s) = 2\int_{\tau=s_1}^{\tau=s}\int_{\sigma=s_1}^{\sigma=\tau} \Psi_{\gamma\delta}^{\nu}(\tau)M_{\beta}^{\delta}(\sigma)\kappa_{\gamma}(s1)\Delta\varphi^{\beta}(\sigma,q(\sigma))\,d\sigma d\tau.$$

Comparison of (26.10) with (23.3) shows that, formally, (26.10) is a special case of (23.3) with $L_{\alpha\gamma}^{\nu}(\tau)$ replaced by $\Psi_{\gamma\delta}^{\nu}(\tau)$ and $\Delta\varphi^{\alpha}(\sigma,\,q(\sigma))$ replaced by $\kappa_{\gamma}(s1)$. Hence

by the argument which established (23.16), we conclude that for each ν and each ordered pair of indices (γ , β) the corresponding term in the right member of (26.10) is the double Stieltjes integral

$$(26.11) \qquad \kappa_{(\gamma)}(s_1) \iint_S \Delta\varphi^{(\beta)}(\sigma, q(\sigma))\, dg^{\nu I 2}_{(\gamma)(\beta)}(s_1;\sigma,\tau)$$

in which the integrator function is

$$(26.12) \qquad g^{\nu I 2}_{\gamma\beta}(s_1;\sigma,\tau) \equiv \int_{v=s_1}^{v=\tau} \int_{u=s_1}^{u=\sigma} \Psi^{\nu}_{\gamma\delta}(\tau)\, M^{\delta}_{\beta}(\sigma)\, d\sigma\, d\tau.$$

[See (23.5).]

Evidently, when we extend the summation convention also to γ , β , it follows from (26.11) that (26.10) can be written as

$$(26.13) \qquad \mu^{I2}_{\nu}(s) = \kappa_{\gamma}(s_1) \iint_S \Delta\varphi^{\beta}(\sigma, q(\sigma))\, dg^{\nu I 2}_{\gamma\beta}(s_1;\sigma,\tau).$$

As we observed in § 23, the functions $\mu^{II1}_{\nu}(s)$ defined by (23.2) are, formally, like those designated $\mu^{III}_{\nu}(s)$ in (21.4), in that only a single quadrature is involved. By comparing (23.2) with (24.2), which is equivalent to (21.4), we see that (23.2) is a special case of (24.2) with $L^{\nu}_{\alpha\gamma}(\tau)$ in place of $N^{\nu}_{\alpha\beta}(\tau)$ and with

$K_\gamma(s_1)$ replacing $\Delta\varphi^\beta(\tau, q(\tau))$. Thus, if we define [see (24.5)]

$$(26.14) \quad g^{\nu II_1}_{\alpha\gamma}(s_1; \sigma, \tau) \equiv \int_{v=s_1}^{v=\min(\sigma,\tau)} \cdot \quad L^\nu_{\alpha\gamma}(v)\, dv ,$$

it follows by the argument which established the equivalence of (24.3) and (24.4) that, for each ordered pair of the indices (α , γ) the corresponding term in the right member of (23.2) is the double Stieltjes integral

$$(26.15) \quad K_{(\gamma)}(s_1) \iint_S \Delta\varphi^{(\alpha)}(\tau, q(\tau))\, dg^{\nu II_1}_{(\alpha,\gamma)}(s_1; \sigma, \tau).$$

Furthermore, extension of the summation convention to α , γ enables us, therefore, to write (23.2) in the form

$$(26.16) \quad \mu^{II_1}_\nu(s) \equiv K_\gamma(s_1) \iint_S \Delta\varphi^\alpha(\tau, q(\tau))\, dg^{\nu II_1}_{\alpha\gamma}(s_1; \sigma, \tau).$$

Finally, we note that the functions $\mu^{IV_1}_\nu(s)$ defined in (25.2) are formally like the functions $\mu^{III}_\nu(s)$ in (24.2). Given ν , for each j (j = 1, . . . , n), the function $F^\nu_{\alpha j\gamma}(\tau)$ in (25.2) corresponds to $N^\nu_{\alpha\beta}(\tau)$ in (24.2), the function $\Delta\varphi^\alpha_{qj}(\tau, q(\tau))$ to $\Delta\varphi^\alpha(\tau, q(\tau))$, and the constant $K_\gamma(s_1)$ to the function $\Delta\varphi^\beta(\tau, q(\tau))$. Consequently, the argument by which we showed that (24.3) is expressible as the double Stieltjes integral (24.4) enables

us to conclude that, given ν, for each ordered pair of the indices α, γ, <u>and</u> <u>for</u> <u>each</u> j, the sum of the corresponding terms in the right member of (25.2) is the double Stieltjes integral

$$(26.17) \quad K_{(\gamma)}(s_1) \iint_S \Delta\varphi^{(\alpha)}_{q^{(j)}}(\tau, q(\tau))\, dg^{\nu\ \mathrm{IV}_1}_{(\alpha)(\gamma)(j)}(s_1; \sigma, \tau)$$

in which the integrator function is [See (24.5)]

$$(26.18) \quad g^{\nu\ \mathrm{IV}_1}_{\alpha\gamma j}(s_1; \sigma, \tau) = \int_{v=s_1}^{v=\min(\sigma,\tau)} P^{\nu}_{\alpha j \gamma}(v)\, dv .$$

Summing on the index j in (26.17) gives us, as the sum of those terms in the right member of (25.2) which correspond to a given ν and an ordered pair of indices (α, γ), the expression

$$(26.19) \quad \mu_\nu^{\alpha\gamma\ \mathrm{IV}_1}(s) = K_{(\gamma)}(s_1) \iint_S \Delta\varphi^{(\alpha)}_{q^j}(\tau, q(\tau))\, dg^{\nu\ \mathrm{IV}_1}_{(\alpha)(\gamma)j}(s_1; \sigma, \tau).$$

The remarks following (25.2) apply equally well to (26.19). Summing on α, γ in (26.19) enables us to write (25.2) as

$$(26.20) \quad \mu_\nu^{\mathrm{IV}_1}(s) = K_\gamma(s_1) \iint_S \Delta\varphi^{\alpha}_{q^j}(\tau, q(\tau))\, dg^{\nu\ \mathrm{IV}_1}_{\alpha\gamma j}(s_1; \sigma, \tau),$$

in which the right member is the sum of nm^2 double Stieltjes integrals, n such integrals for each ordered pair of the indices α, γ.

27. The Functions $\mu_\nu(s)$ Corresponding to a Pair $\Delta\varphi^\alpha, \Delta\varphi^\beta$ of Disturbance Functions.

To compute the second differential effect on a trajectory element - say, on range - due to a disturbance $(\Delta q_0, \Delta\varphi)$, we have seen that one or more of the functions ζ^i defined by (16.1) must be calculated. In (20.15) we have seen, furthermore, that each second variation ζ^i is expressible in terms of the n functions $\xi^i_1, \ldots, \xi^i_n$ appearing in the i^{th} row of the determinant (20.8) and the n functions $\mu_\nu(s)$, $[\nu = 1, \ldots, n]$ defined by (20.13). In §§ 21-26 we have seen how each term in any one of the functions $\mu_\nu(s)$ is expressible as a double Stieltjes integral. Evidently we must add these integrals to obtain, for a given ν, the corresponding μ_ν. It is fortunate that all these integrals have the same region of integration—the square designated J in § 22 and S in § 23. Moreover, many of these integrals have the same integrands but different integrator functions; we found that we could combine those integrals having this property into a single integral. But this combining process is only partially finished; it is our purpose in the present section to complete it.

In order to help us fix in mind what needs to be done, let us suppose, for example, that we wish to

obtain the second differential effect on range of the combined presence of a range-wind $w_x(y)$ and a non-standard temperature $\theta(y)$. Let us further suppose that, among the fourteen disturbance functions $\Delta\varphi^1, \ldots, \Delta\varphi^{14}$, all are identically zero except those which represent $w_x(y)$ and $\theta(y)$; say these are $\Delta\varphi^8$, $\Delta\varphi^9$, respectively. For simplicity, we shall assume also that the initial conditions are standard, so that $\Delta q_0^i = 0$, for $i = 1, \ldots, 6$. Then for each value of ν ($\nu = 1, \ldots, 6$), the corresponding function μ_ν will be the sum of all those terms in the right member of (21.1) which contain $\Delta\varphi^8$, or $\Delta\varphi^9$, or both $\Delta\varphi^8$ and $\Delta\varphi^9$, or the <u>derivatives</u> $\Delta\varphi^8_y$, $\Delta\varphi^9_y$, in their integrands. Such terms will be those whose integrands contain the products $\Delta\varphi^8 \Delta\varphi^9$, $(\Delta\varphi^8)^2$, $(\Delta\varphi^9)^2$, $\Delta\varphi^8 \Delta\varphi^9_y$, $\Delta\varphi^8_y \Delta\varphi^9$, $\Delta\varphi^8 \Delta\varphi^8_y$, and $\Delta\varphi^9 \Delta\varphi^9_y$. To these products there correspond the ordered pairs (8,9), (8,8), (9,8), (9,9) of indices α, β on the disturbance functions. For example, the terms which correspond to the ordered pair (8,9) consist of those containing the product $\Delta\varphi^8_y \Delta\varphi^9$ and <u>half</u> of those containing the product $\Delta\varphi^8 \Delta\varphi^9$; the other half of the latter terms, together with <u>all</u> the terms containing the product $\Delta\varphi^9_y \Delta\varphi^8$ correspond to the ordered pair (9,8). The terms containing $(\Delta\varphi^8)^2$ and $\Delta\varphi^8_y \Delta\varphi^8$ are those which correspond to the ordered pair (8,8), etc. Clearly we shall obtain all the terms which occur in the function μ_ν

if we add together the four collections of terms which correspond to the four ordered pairs of indices (8,9), (8,8), (9,8), and (9,9), respectively. This can be most easily accomplished if we first find an expression for the sum of <u>all</u> the terms in the right member of (20.13) which correspond to a given value of ν and a given ordered pair (α , β) of indices on disturbance functions, initial conditions assumed standard.

The expression we seek is thus the sum of (22.21), (23.16), (24.4) and (25.8). [Since the initial conditions are normal, the quantities $K_\gamma(s_1)$, $K_\delta(s_1)$, defined by (20.12) and occurring in the functions considered in § 26, are all zero. Hence none of the functions $\mu_\nu^{I1}(s)$, etc., discussed in § 26 occurs in the expression we desire.] Since the integrand in each of the expressions (22.21), (23.16), and (25.4) is $\Delta\varphi^{(\alpha)}(\tau, q(\tau))\,\Delta\varphi^{(\beta)}(\sigma, q(\sigma))$ and the region of integration in each is the square S, the sum of these three expressions is

$$(27.1)\quad \iint_S \Delta\varphi^{(\alpha)}(\tau, q(\tau))\,\Delta\varphi^{(\beta)}(\sigma, q(\sigma))\,dg_{(\alpha)(\beta)}(s, s_1; \sigma, \tau),$$

in which the integrator function $g_{(\alpha)(\beta)}(s, s_1; \sigma, \tau)$ is

$$(27.2)\quad g_{(\alpha)(\beta)}(s, s_1; \sigma, \tau) = g^{\nu I}_{(\alpha)(\beta)} + g^{\nu II}_{(\alpha)(\beta)} + g^{\nu III}_{(\alpha)(\beta)}.$$

The integrands of the n integrals in the right member of (25.8) are dependent upon j and are conceivably different

for different values of j. Thus, in general, we are not able to combine the integral in (27.1) with those in (25.8), to obtain in the form of one double Stieltjes integral the sum of all the terms in the right member of (21.1) which correspond to an ordered pair of indices. But, as we observed in the remark following (25.9), for trajectories along which it is possible to use altitude y as independent variable and when the disturbances depend only upon altitude, then the integrals (25.8) are identically zero.

Consequently, in the case of the illustrative example considered earlier in this section (involving the two disturbance functions $\Delta\varphi^8$, $\Delta\varphi^9$ dependent on y alone along a one-branched trajectory, with standard initial conditions), for each ν ($\nu = 1, \ldots, 6$) the function $\mu_\nu(s)$ is the sum of the four double Stieltjes integrals obtained from (27.1) by giving (α, β) the pairs of values (8,8), (8,9), (9,8), and (9,9). Of course, the two integrals corresponding to the second and third of these pairs of indices are identical.

If initial conditions are not standard, then the integrals considered in § 26 must also be computed, and added to any obtained by the use of (27.1).

Chapter VII

McSHANE'S THEOREM ON THE (FIRST) DIFFERENTIAL. AN ANALOGOUS THEOREM ON THE SECOND DIFFERENTIAL.

28. One-branched Trajectories. Uniqueness of Solutions of Differential Equations of Motion.

Our theorem (15.10) on the existence of the second differential of the mapping defined by (7.1), (7.2) is a result of the same type as that of Bliss, (11.3), on the first differential. The similarity arises from the fact that, in both theorems, the norm of the disturbance includes the derivative of the disturbance functions with respect to the variable y when the disturbance is one (such as wind, abnormal temperature, or abnormal density) which is assumed dependent only on altitude y. As pointed out in the paragraph preceding (11.3), McShane's main theorem [(11.7) of [McS]] is an improvement on (11.3) in the sense that it establishes the existence of the first differential when N_{1y}, from which the objectionable derivative is missing, is used as norm. Our chief objective in this chapter will be to establish a theorem on the second differential analogous to that of McShane on the first differential, and thus to sharpen our conclusion (15.10) in a similar way to that in which McShane's theorem improves on that of Bliss. But to accomplish this objective we have to introduce a new norm,

which can be larger than N_{gy} and which may not reduce to N_0 even when the disturbance depends only on altitude. For this new norm includes the values of the first partial derivatives of the disturbance in a neighborhood of the summit; and the derivative of, say, range-wind $w_x(y)$ with respect to y is not necessarily zero near the summit.

In order to provide the background necessary for stating McShane's theorem, and also to assemble for the reader's convenience certain material from [McS] which is essential to the proof of our theorem on the second differential, we shall again quote several passages from that paper in this chapter. We begin with certain remarks on one-branched trajectories, leading to a theorem on the continuity and uniqueness of the solutions of the differential equations of motion (3.3). The excerpts which we quote in this section are taken from §§7, 8, 9, and 10 of the manuscript of [McS].

"In § 1 (of [McS]; § 2 of the present paper) it was pointed out that for the purposes of ballistics it was unobjectionable to include, in the definition of the norm, the derivatives of the disturbances with respect to the variables v_x, v_y, v_z, x, and z; it is the inclusion of derivatives with respect to y that we desire to avoid. In the definition of N_s the derivatives of φ^α with respect to the q^j are present, but the derivative with respect to s is absent. Thus if we identify s with y the situation

is what we are seeking. Since $s = y$, the norm N_s will be written as N_y. Suppose then that we are interested only in the subset of trajectories with $v_y > 0$. On these we can introduce y as independent variable and apply theorems" (7.8), (10.1), and (11.3). "In fact, if we are interested only in the effects of changes of initial conditions, wind, density, and temperature, the latter being regarded as a function of y alone, ΔF is identically zero and $\Delta\varphi$ is independent of q^j, so $N_y = N_0$. The norm N_y does not then merely lack the objectionable property of N_s when s is t or x or anything but y; it has coincided with the very reasonable norm N_0. The same situation prevails if we regard only the trajectories with $v_y < 0$; these include dive-bombing trajectories, and level-bombing trajectories if we cut off the short time at the top during which the bomb is sheltered in the bomb-bay.

"However, for many trajectories, including all ordinary ground-to-ground artillery trajectories, it is impossible to use y as independent variable, because the trajectory has an ascending and a descending branch. We can use the time t as independent variable. But if ρ, θ, w_x, w_z are merely continuous functions of y, the standard theorems on differential equations will not even guarantee the uniqueness of the solution. Our first task, then, is to show that even when ρ, etc., are merely continuous in y, the equations of motion are uniquely solvable."

McShane's proof of the continuity and uniqueness of the solutions of the system (6.3) makes strong use of a lemma, (8.4) of [McS], concerning the propagation of disturbances along the descending branch of a trajectory. This same lemma will also be employed in our proof of the principal theorem of this chapter on the second differential. We state it in the next excerpt from [McS], after first quoting some introductory passages.

"Let us suppose that we desire to solve" (6.3) "with certain initial position and velocity, the density ρ, temperature θ and wind components w_x, w_z being continuous except perhaps at a finite number of values of y, where they may have simple jump discontinuities, and having bounded partial derivatives with respect to x and z continuous except at finitely many values of y. If $v_{yo} > 0$, up to the summit we can introduce y as independent variable in" (6.3) "and find that the resulting equations are uniquely solvable. Likewise if two trajectories have coincident x, t, z, v_x, v_y, v_z at some level y on the descending branch, these descending branches are identical. However, these standard theorems do not at once inform us that a given ascending branch might not be followed by two distinct descending branches. By continuity, the values of x, y, z, v_x, $v_y = 0$, v_z are uniquely determined at the summit because the ascending branch is unique. We must show that these summital values determine a unique descending branch.

"Let us first transform equations" (6.3). "Suppose that we are given the descending branch of a trajectory, corresponding to times t in an interval $[t_o, t_2]$, and defined by functions $x(t), \ldots, v_z(t)$. This we shall call the 'original trajectory'. We understand that t_o corresponds to the summit. Let $\bar{x}(t), \ldots, \bar{v}_z(t)$, $\bar{t}_3 \leqq t \leqq \bar{t}_4$, be another arc of trajectory (the 'second trajectory') with $\bar{v}_y < 0$ and with $\bar{y}$ between $y(t_o)$ and $y(t_2)$. To each time t between $\bar{t}_3$ and $\bar{t}_4$ there corresponds a time s between t_o and t_2 for which

(28.1) $$y(s) = \bar{y}(t).$$

If in particular s_3, s_4 correspond to $\bar{t}_3$, $\bar{t}_4$ respectively, equation" (28.1) "establishes a one-to-one correspondence between s and t, which we can write in either of the forms

(28.2i) $$s = s(t) \quad (\bar{t}_3 \leqq t \leqq \bar{t}_4),$$

(28.2ii) $$t = t(s) \quad (s_3 \leqq s \leqq s_4).$$

This new variable s we introduce as independent variable on the second trajectory."

There follow in [McS] the details of the transformation of (6.3). We do not give them here as they are not necessary in what follows. The result is a system of equations in which s is independent variable, from which

y is absent, and which is satisfied by the "second trajectory". [When y is needed, it can be obtained from (18.1).] By introducing the symbols $q^1, \ldots, q^6$ for the variables v_y, v_x, v_z, x, z, t in that order, this system can be written in a form analogous to (7.1), and the "original trajectory" satisfies a similar system. In terms of the new independent variable s and the new dependent variables q^i defined above we can now state McShane's lemma on the propagation of disturbances along the descending branch.

(28.3) Lemma [(8.4) of [McS]]: Corresponding to the descending branch $q^i = q^i(s)$, $(t_0 \leqq s \leqq t_2)$, of the original trajectory there is a constant K_0 with the following property. If $q_1(s)$ and $q_2(s)$, $(s_3 \leqq s \leqq s_4)$, are descending arcs of trajectories with $t_0 < s_3 \leqq s_4 \leqq t_2$, both being computed with the same functions F^i, a_y, φ^* but with possibly different initial values, and if

(28.3i) $$q_1^1(s)/q^1(s) \geqq 1/2, \quad q_2^1(s)/q^1(s) \geqq 1/2, \quad (s_3 \leqq s \leqq s_4),$$

then

(28.3ii) $$|q_2(s)-q_1(s)| \leqq K_0|q_2(s_3)-q_1(s_3)|, \quad (s_3 \leqq s \leqq s_4).$$

McShane used the lemma (28.3) to show the uniqueness of the solutions of (6.3); i.e., that two trajectories

satisfying the system (6.3) and having the same initial values $v_{yo} = 0$, v_{xo}, v_{zo}, x_o, z_o, t_o, y_o are identical. Together with (7.8) it was also used to prove the continuity of the solutions of (6.3) as functions of the disturbances (§ 10 of [McS]). More specifically, the latter property was established in the sense described in the following passage from § 10 of [McS]:

"Now we assume:

(28.4) The function E is a continuously differentiable function of x, y, z, v_x, v_y, v_z, w_x, w_z, ρ, θ regarded as its independent variables.

(28.5) The functions w_x, w_z, ρ, θ, a_x, a_y, a_z, and their first- order partial derivatives with respect to all variables except y, are bounded and are continuous except perhaps at a finite number of values of y, where they may have simple jump discontinuities.

(28.6) "Let (28.4) and (28.5) be satisfied. Let the original initial values, and the wind, density, temperature, and added accelerations be such that the solution of (6.3) lies in R_o for $t_1 \leqq t \leqq t_2$. Then for all other initial values, etc., satisfying (28.4) and (28.5) and lying sufficiently close to the original values in the sense of the norm N_o, equations (6.3) will again have a unique solution,

and this will lie in R_o for $t_1 \leq t \leq t_2$, and as K_o tends to zero the change in the solutions (with the usual norm (7.6)) will also tend to zero. In this theorem we may replace equations (6.3) by any equivalent system."

29. McShane's Theorem on the First Differential.

In this section, we shall state McShane's theorem [(11.7) of [McS]] on the existence of the first differential when the norm of the disturbance is N_{1y} (N_{1s} when $s = y$); we shall also describe in detail the new coordinate system introduced by McShane in the proof of his theorem, for we shall use it in the proof of our theorem on the second differential in § 31. Quotations occurring below are from the original manuscript of [McS].

We shall assume that we have given a two-branched trajectory T satisfying the system (6.3). We suppose that T has a summit at the point which corresponds to a value t_o of t between t_1 and t_2. As an additional hypothesis we also assume:

(29.1) "On a neighborhood of the summital values $(x(t_o), y(t_o), \ldots, v_z(t_o))$ all eleven functions φ^α are continuous and continuously differentiable; and the F^i are everywhere twice continuously differentiable.

"In practice this hypothesis is quote harmless. The 'normal' trajectory, from which disturbances are measured, is always computed with constant w_x and w_z (ordinarily both are identically zero) and with ρ and θ dependent on y alone, and either analytic or at worst composed of two analytic arcs, meeting (with discontinuous derivatives) at the lowest altitude of the stratosphere."

Although T is a <u>two-branched</u> trajectory, we use the <u>norm</u> appropriate for <u>one-branched</u> trajectories. This fact cannot be made too emphatic, for the crux of McShane's theorem is that the norm N_{1y} <u>can</u> be used in the proof of the existence of a first differential on a <u>two-branched</u> trajectory; we are <u>not</u> limited to the use of the less desirable norm N_{1t}, despite our inability to use y as independent variable along a two-branched trajectory. (As an "extra dividend", of course, the norm N_{1y} reduces to the more reasonable norm N_0 when the disturbance depends only on altitude.)

Suppose then that we have an original trajectory T lying in R_0 and satisfying (29.1). If Δx_0, Δy_0, Δz_0, Δv_{x0}, Δv_{y0}, Δv_{z0}, Δt_0, $\Delta \rho$, $\Delta \theta$, Δw_x, Δw_z are disturbances with sufficiently small norm N_0, then the equations of motion with initial conditions $\bar{x}_0 = x_0 + \Delta x_0$, etc., with density, etc., $\bar{\rho} = \rho + \Delta \rho$, $\bar{\theta} = \theta + \Delta \theta$, $\bar{w}_x = w_x + \Delta w_x$, $\bar{w}_z = w_z + \Delta w_z$, will have solutions likewise defined on $[t_1, t_2]$. We now choose a value t_3 between

t_0 and t_2, and between $y(t_3)$ and $y(t_2)$ change to y as independent variable. We find that along the new trajectory $\overline{T}$ the variables x, etc., differ from the original values x, etc., by the quantities

$$\Delta x(y) = \overline{x}(y) - x(y), \qquad \Delta z(y) = \overline{z}(y) - z(y),$$
$$\Delta t(y) = \overline{t}(y) - t(y), \qquad \Delta v_x(y) = \overline{v}_x(y) - v_x(y),$$
$$\Delta v_y(y) = \overline{v}_y(y) - v_y(y), \qquad \Delta v_z(y) = \overline{v}_z(y) - v_z(y).$$

We now state McShane's theorem on the existence of a first differential.

(29.2) Theorem (McShane): Let T be an original trajectory on which (29.1) is satisfied. Let the summit correspond to t_0, and let t_3 be any number such that $t_0 < t_3 < t_2$. Then to each set of disturbances $\Delta x_0, \ldots, \Delta w_z$ there corresponds a set of six functions

$$\eta^i(y) = \eta^i(y; \Delta x_0, \ldots, \Delta w_z)$$

$(i = 1, \ldots, 6)$, $y(t_2) \leqq y \leqq y(t_3)$, with the following properties. They are continuous on $[y(t_2), y(t_3)]$. With the usual norm (7.6), they are linear functions of the disturbances, and are continuous not only when the disturbances are measured by N_{1y}, but even when they are measured by N_0. If we assign v_y, v_x, v_z, x, z, t the alternative symbols $q^1, \ldots, q^6$, respectively, $\eta(y)$

is the differential of [illegible] in the sense that to each positive ϵ there corresponds a positive $\delta(\epsilon)$ such that whenever the disturbances $\Delta q_0, \ldots, \Delta [illegible]$ [illegible] and [illegible] norm $N_{1p} < \delta(\epsilon)$ the inequality

$$|\Delta q(y) - Dq(y)| \leq \epsilon N_{[illegible]} = [illegible] N_{1p}$$

$$| y(t_0) \leq y \leq y(t_0) |$$ is satisfied.

The coordinate system employed by [illegible] in proving (29.3) is described in the following paragraphs quoted from the manuscript of [illegible].

"There is no loss of generality in assuming that the axes (chosen in § [illegible]) are translated horizontally so that the summit is on the y-axis and are rotated about the y-axis so that at the summit the velocity vector is $(v_x(t_0), 0, 0)$, with $v_x(t_0) > 0$. For simplicity we shall assume that the trajectory does not intersect the yz-plane except at the summit. This is convenient rather than essential, but it hardly seems worth complicating the notation merely to include differential effects on trajectories of boomerangs.

"Now we construct a new coordinate system in (x, y, z, v_x, v_y, v_z)-space. The last four of these are unchanged; the first two are replaced by new variables [illegible] as follows. We choose a positive number [illegible] (standing for "clearance") and in the yz-plane we construct a [illegible]

line at altitude $y(t_0) - \kappa$. This line is the upper boundary of the half-plane

(29.3) W; $x = 0$, $y = y(t_0) - \kappa$.

Let P be a point not on W, and let (x, y, z, v_x, v_y, v_z) be its coordinates. We define p to be the distance from P to W. If $y \leqq y(t_0) - \kappa$ we pass through P a horizontal half-plane bounded by W. If $y > y(t_0) - \kappa$ we pass through P a half-plane bounded by the upper boundary of W. In either case the half-plane through P intersects the original trajectory in exactly one point; this is rather obvious, and will be proved incidentally in the sentences following (29.8). To this point corresponds a value of the time on the original trajectory. This time we denote by the symbol s. Thus if $y \leqq y(t_0) - \kappa$ and $x < 0$, to P there corresponds one value of s, and this s is uniquely determined by y, and is less than t_0. If $y \leqq y(t_0) - \kappa$ and $x > 0$, to P there corresponds one value of s, and this s is uniquely determined by y, and is greater than t_0. Where $y > y(t_0) - \kappa$, the two variables s,p are uniquely determined by (x,y) and conversely; but y alone does not determine s, nor does s alone determine y.

There is a time at which y(t) on the ascending branch of the original trajectory is equal to $y(t_0) - \kappa$; this time we denote by s_a. There is a time at which y(t)

on the descending branch of the original trajectory is equal to $y(t_0) - h\tau$; this time we denote by s_d.

"Now let $\overline{T}$ be another trajectory (defined by functions $\bar{x}(s)$, $\bar{y}(s)$, etc.) having a single point in common with the yz-plane, at which point $v_x > 0$ and $y > y(t_0) - h\tau$. On the ascending branch up to height $y(t_0) - h\tau$, s is uniquely determined by $\bar{y}$ and conversely, and $x = -p$. Let $\bar{t}(s)$ be the time at which $\bar{y} = y(s)$. Then

$$d\bar{y}(\bar{t}(s))/ds = dy(s)/ds,$$

or

$$\bar{v}_y \, d\bar{t}/ds = v_y(s).$$

Hence

$$d\bar{t}/ds = v_y(s)/\bar{v}_y. \tag{29.4}$$

The equations of motion become

$$\left\{\begin{aligned} d\bar{v}_y/ds &= -E\bar{u}_y v_y/\bar{v}_y - (g-a_y)v_y/\bar{v}_y, \\ d\bar{v}_x/ds &= (-E\bar{u}_x + a_x)v_y(s)/\bar{v}_y, \\ d\bar{p}/ds &= [\bar{v}_x v_y(s)/\bar{v}_y]\,\mathrm{sgn}(s-t_0), \\ d\bar{v}_z/ds &= (-E\bar{u}_z + a_z)v_y(s)/\bar{v}_y, \\ d\bar{z}/ds &= \bar{v}_z v_y(s)/\bar{v}_y, \\ d\bar{t}/ds &= v_y(s)/\bar{v}_y. \end{aligned}\right. \tag{29.5}$$

In the right members the functions E are to be computed as functions of $\bar{x}$, etc. along $\overline{T}$, with $\bar{x}$, $\bar{y}$ replaced by $-p$ and $y(s)$ respectively. A similar argument applies to the part

of the descending branch with $\overline{y} \leq y(t_0) - \kappa$; the only difference is that $\overline{x}$ is to be replaced by $+p$ instead of $-p$, which also accounts for the factor sgn $(s-t_0)$ in the third of the equations (29.5).

"There remains the portion of $\overline{T}$ with $\overline{y} > y(t_0)-\kappa$. As an auxiliary, we use polar coordinates (p, ϑ) to replace (x,y). The angle θ is uniquely determined by s, and

$$(29.6)\qquad \cos\theta(s) = x(s)/\left[x(s)^2 + \{y(s)-y(t_0)+\kappa\}^2\right]^{1/2}$$
$$\sin\theta(s) = y(s)/\left[x(s)^2 + \{y(s)-y(t_0)+\kappa\}^2\right]^{1/2}.$$

At time $\overline{t}$ at which $\overline{y} > y(t_0) - \kappa$, the rectangular coordinates $(\overline{x},\overline{y})$ transform into $(\overline{p},\overline{\theta})$, where

$$(29.7)\qquad \overline{\theta}(\overline{t}) = \overline{\theta}(s(\overline{t})).$$

By elementary computations we find that

$$(29.8)\qquad \begin{aligned} d\overline{p}/d\overline{t} &= \overline{v}_x \cos\overline{\theta} + \overline{v}_y \sin\overline{\theta}, \\ d\overline{\theta}/d\overline{t} &= -(\overline{v}_x/\overline{p}) \sin\overline{\theta} + (\overline{v}_y/\overline{p}) \cos\overline{\theta}. \end{aligned}$$

By applying the same argument to T, we find that the second of equations (29.8) is valid if the dashes are omitted and $\overline{t}$ replaced by s. We now add the assumption (retained throughout the rest of the proof) that κ is small enough so that v_x remains positive between s_a and s_d; later we shall reduce κ still further. Then $(v_x/p)\sin\theta$ is positive for $s_a < s < s_d$. For $s < t_0$ we know that v_y is negative, while for $s > t_0$ it is positive. At $s = t_0$,

$\cos \theta$ (s) changes from negative to positive. So $(v_y/p) \cos \theta$ is negative except at to. Thus $d\theta/ds$ is negative for $s_a \leq s \leq s_d$, and so s is a single-valued function of θ for $-\pi \leq \theta \leq 0$.

"Furthermore, by making $\bar{T}$ sufficiently close to T in the usual norm (that is, by making the norm N_0 of the disturbances small), we can make the right members of (29.8) differ from the corresponding expressions for T by an amount less than half the minimum of $(v_x/p)\sin\theta-(v_y/p)\cos\theta$, so that the right member of the second equation is surely negative. Then $\bar{\theta}$ $(\bar{t})$ has a single-valued differentiable inverse $\bar{t}(\bar{\theta})$. By (29.7),

$$(29.9) \quad d\bar{t}/ds = [d\,\theta(s)/ds] \big/ [d\,\bar{\theta}(\bar{t})/dt]$$

$$= \frac{-v_x(s)\bar{p}\,\sin\theta(s) + v_y(s)\bar{p}\,\cos\theta(s)}{-\bar{v}_x p(s)\,\sin\bar{\theta} + \bar{v}_y\,p(s)\,\cos\bar{\theta}}\,.$$

The equations of motion for $s_a \leq s \leq s_d$ now become

$$(29.10)\left\{\begin{aligned}
d\bar{v}_y/ds &= (-E\bar{u}_y - g + a_y)\,d\bar{t}/ds,\\
d\bar{v}_x/ds &= (-E\bar{u}_x + a_x)\,d\bar{t}/ds,\\
d\bar{p}/ds &= [\,\bar{v}_x \cos\theta(s) + \bar{v}_y \sin\theta(s)\,]\,d\bar{t}/ds,\\
d\bar{v}_z/ds &= (-E\bar{u}_z + a_z)\,d\bar{t}/ds,\\
d\bar{z}/ds &= \bar{v}_z\,d\bar{t}/ds,\\
d\bar{t}/ds &= d\bar{t}/ds,
\end{aligned}\right.$$

where in the right members $d\bar{t}/ds$ is to be replaced by its value from (29.9), $\bar{y}$ by $\bar{p} \sin \theta(s) + y(t_0) - k\tau$, $\bar{x}$ by $\bar{p} \cos \theta(s)$, and the sine and cosine of $\theta(s)$ are in turn to be replaced by their values from (29.6)."

30. Another New Norm.

Theorem (29.2) leads us to conjecture that the existence of the second differential of the mapping (6.1) could be established with the use of the norm N_{2y}. But an example which will be given in § 32 shows that this is not possible.

We shall, however, demonstrate the existence of the second differential relative to a new norm—presently to be defined and to be designated as $\overset{k}{N}_{2y}$. This new norm is not as objectionable as the norm N_{2t}; for it does not, as does N_{2t}, include the absolute values of the first and second partial derivatives of the disturbance functions with respect to altitude y at every point of R_0. On the other hand, it is not as desirable a norm as $\mathcal{N}_{2y}$, because it does include (whereas N_{2y} does not include) the absolute values of the first—but not the second—derivatives of the disturbance functions with respect to altitude at points of R_ρ which lie in an arbitrarily small neighborhood of the summit.

The new norm we define as follows:

(30.1) Definition: Let k be any positive number, and let U_k denote the k-neighborhood [in Euclidean 7-space] of the summit $(x_o, y_o, z_o, t_o, v_{xo}, v_{yo}, v_{zo})$. For any disturbance

$$\bar{\Delta} = (\Delta x_o, \Delta y_o, \Delta z_o, \Delta t_o, \Delta v_{xo}, \Delta v_{yo}, \Delta v_{zo};$$
$$\Delta \rho, \Delta \vartheta, \Delta w_x, \Delta w_z)$$

let $N^k_{2y}(\bar{\Delta})$ be the larger of the two numbers:

(i) $\mathcal{N}_{2y}(\bar{\Delta})$;

(ii) $\sup \left[(\Delta \rho_{q^i})^2 + (\Delta \vartheta_{q^i})^2 + (\Delta w_{x_{q^i}})^2 + (\Delta w_{z_{q^i}})^2 \right]^{1/2}$,

for all q^i of R_o in U_k, in which q^i is replaced by $x, y, z, t, v_x, v_y, v_z$ in turn.

31. Existence of the Second Differential When the Norm is N^k_{2y}.

We now state our theorem on the second differential comparable to that of McShane on the first differential, (29.2).

(31.1) Theorem: Let T be an original trajectory on which (29.1) is satisfied. Let the summit of T correspond to $t = t_o$, and let t_3 be any number such that $t_o < t_3 < t_2$. Then to each disturbance

$$\bar{\Delta} = (\Delta x_o, \Delta y_o, \Delta z_o, \Delta t_o, \Delta v_{xo}, \Delta v_{yo}, \Delta v_{zo}; \Delta \rho, \Delta \theta, \Delta w_x, \Delta w_z)$$

there correspond two sets of six functions,

$$\eta^i(y) = \eta^i(y\,;\bar{\Delta})$$

and

$$\zeta^i(y) = \zeta^i(y;\bar{\Delta},\bar{\Delta}),$$

$(i = 1, \ldots, 6)$, $y(t_2) \leq y \leq y(t_3)$, with the following properties. The functions $\eta^i(y)$ are the same as those so designated in (29.2) and hence have the properties stated in the conclusion of that theorem. With the usual norm (7.6), the $\zeta^i(y)$ are quadratic functions of the disturbance $\bar{\Delta}$ and are continuous on the interval $[y(t_2), y(t_3)]$ when $\bar{\Delta}$ is measured by the norm $N_{1y}(\bar{\Delta})$. If we assign to v_y, v_x, v_z, x, z, t the alternative symbols $q^1, \ldots, q^6$, respectively, then $\zeta(y)$ is the second differential of q in the sense that to each positive ϵ there corresponds a positive $\delta(\epsilon)$ such that, whenever $\bar{\Delta}$ belongs to ϕ_{yy} and has norm $N_{2y} < \delta(\epsilon)$, the inequality

$$(31.2) \qquad |\Delta q(y) - \eta(y) - (1/2)\,\zeta(y)| \leq \epsilon N_{1y} N_{2y}^k,$$
$$[y(t_2) \leq y \leq y(t_3)],$$

is satisfied.

The quadratic character and continuity property of the functions $\zeta^i(y)$ are consequences of (16.1). In order to establish (31.2), we require first a somewhat more precise estimate for the constant K_2 in (11.3i).

In his proof of (11.3) McShane defines $e^i(s) \equiv \Delta q^i(s) - \eta^i(s)$ [not the same functions as those in (20.1)] and then deduces the inequality

(31.3) $d|e(s)|/ds \leqq A(s)|e(s)| + K_0 N_0(\Delta q_0, \Delta\varphi) N_{1s}(\Delta q_\rho, \Delta\varphi)$,

valid whenever $|e(s)| > 0$, in which K is constant and $A(s)$ is the least upper bound of $F^i_{r\alpha}(\varphi(s,q(s)))\,\varphi^\alpha_{qj}(s,q(s))u^i u^j$ for unit vectors u. He then states that lemma (9.3), applied to (31.3) shows the existence of a constant K_2 such that $|e(s)| \leqq K_0 N_0(\Delta q_0, \Delta\varphi)\, N_{1s}(\Delta q_0, \Delta\varphi)$, which is equivalent to (11.3i). The details of this application of (9.3) are omitted in [McS]. Upon supplying them, we find that

(31.4) $|e(s)| \leqq \exp\int_{s_1}^{s} A(t)\,dt \left\{ |e(s_1)| + \int_{s_1}^{s} K N_0 N_{1s}\left[\exp\int_t^{s_1} A(\tau)\,d\tau\right] dt \right\}$,

in which we have written N_0 and N_{1s} for the first and second norms, respectively, occurring in the right member of (31.3). But $e^i(s_1) \equiv 0$, so (31.4) reduces to

(31.5) $|\Delta q(s) - \eta(s)| \leqq K\left[\int_{s_1}^{s}\left\{\exp\int_t A(\tau)\,d\tau\right\} dt\right] N_0 N_{1s}$.

On the original trajectory, s and t are identical. So the equations (29.10) hold when the dashes are omitted and dt/ds is replaced by unity. Let k be a positive number, and let U_k be the k-neighborhood of the summit. By the

hypothesis (29.1), returning to the $F(\varphi(s,q)$ notation in (29.10), for all q in a neighborhood U of the summital values $q(t_0)$ the functions φ_{qi} and $F_{r\alpha}(\varphi(s,q))$ remain bounded. Clearly we lose no generality by assuming $U \subset U_k$. Then there exists a positive number κ_0 so small that on the corresponding interval $s_{a0} \leqq s \leqq s_{d0}$ the quantities q(s) on the original trajectory remain in U. Furthermore, we suppose that the norm N_0 of the disturbance is small enough so that the quantities $\overline{q}(s)$ on the disturbed trajectory also remain in U. Let A be an upper bound for A(s) on U. Then for each positive $\kappa \leqq \kappa_0$ and the corresponding s_a and s_d determined by κ, it follows from (31.5) that there exists a constant K_4 determined by A, K, and κ_0 such that

$$(31.6) \qquad |\Delta q(s_d) - \eta(s_d)| \leqq K_4 (s_d - s_a) N_0 N_{1s}.$$

Now let ϵ be any positive number. Let K_0 be the positive number whose existence is assured by (28.3). We now choose a positive number κ small enough to satisfy $\kappa \leqq \kappa_0$ and $\kappa \leqq y(t_0) - y(t_3)$, as well as the additional requirement

$$(31.7) \qquad s_d - s_a \leqq \epsilon/4\, K_0 K_4.$$

(This is possible, since the left member of (31.7) tends to zero with κ.) If $(\Delta q_0, \Delta\varphi)$ is a disturbance, we define $\Delta_1\varphi(s,q)$ to be $\Delta\varphi(s,q(s))$ for $s_a \leqq s \leqq s_d$ and

to be $\Delta\varphi(s,q)$ otherwise. Let $\Delta_2\varphi = \Delta\varphi - \Delta_1\varphi$, so that

$$(31.8)\quad (\Delta q_0, \Delta\varphi) = (\Delta q_0, \Delta_1\varphi) + (0, \Delta_2\varphi).$$

On the original trajectory T the disturbances $\Delta\varphi$ and $\Delta_1\varphi$ are identical. Hence [Cf. (8.8)]

$$(31.9)\quad \eta(s;q_0,\varphi;\Delta q_0, \Delta_1\varphi) \equiv \eta(s;q_0,\varphi;\Delta q_0,\Delta\varphi)$$

and

$$(31.10)\quad \zeta(s;\Delta_1\varphi, \Delta_1\varphi) \equiv \zeta(s;\Delta\varphi,\Delta\varphi)$$

for $t_3 \leqq s \leqq t_2$. From the definition of $\Delta_1\varphi$, we obviously have

$$(31.11)\quad N_0(\Delta q_0, \Delta_1\varphi) \leqq N_0(\Delta q_0, \Delta\varphi).$$

From (31.8), (31.11), and the triangle inequality it follows that

$$(31.12)\quad N_0(0, \Delta_2\varphi) \leqq 2N_0(\Delta q_0,\Delta\varphi).$$

On the interval $s_a \leqq s \leqq s_d$ the functions $\Delta_1\varphi$ are dependent on s alone, so they contribute nothing in (11.2ii) or in (12.7). Thus

$$(31.13)\quad \sup|\Delta_1\varphi_{qj}(s,q)| \quad (t_1 \leqq s \leqq t_2)$$

$$= \sup|\Delta\varphi_{qj}(s,q)| \quad (t_1 \leqq s \leqq s_a \text{ and } s_d \leqq s \leqq t_2),$$

and

$$(31.14)\quad \sup |\Delta_1 \varphi_{q^j q^k}(s,q)| \quad (t_1 \leqq s \leqq t_2)$$
$$= \sup |\Delta \varphi_{q^j q^k}(s,q)|$$
$$(t_1 \leqq s \leqq s_a, \text{ and } s_d \leqq s \leqq t_2).$$

But on the intervals last indicated the variables q^j are all the variables $v_y, \ldots, t$ <u>except</u> y. Hence the quantities in (31.13) and (31.14) cannot exceed the supremum of the partials of the $\Delta\varphi$ with respect to all the variables except y. From this and (31.11) we see that

$$(31.15)\quad N_{1s}(\Delta q_0, \Delta_1\varphi) \leqq N_{1y}(\Delta q_0, \Delta\varphi)$$

and

$$(31.16)\quad N_{2s}(\Delta q_0, \Delta_1\varphi) \leqq N_{2y}(\Delta q_0, \Delta\varphi)$$

Besides the original trajectory T and the disturbed trajectory $\bar{T}$ we shall also consider a trajectory T_1: $q = q_1(s)$ computed with the same initial values $q_0 + \Delta q_0$ as those of $\bar{T}$ but with functions $\varphi + \Delta_1\varphi$ instead of $\varphi + \Delta\varphi$. Thus we may consider that $\bar{T}$ is obtained from T_1 by applying the disturbance $(0, \Delta_2\varphi)$.

On the original trajectory T we have $q^1(s) = v_y(s) < 0$, $(s_d \leqq s \leqq t_2)$. Hence on this interval $-q^1$ has a positive lower bound, which we denote by $\inf(-q^1)$. By applying the continuity theorem (28.6) to the equations (29.5), (29.10), we see that there exists a positive δ_1 such that if

(31.17) $N_o(\Delta q_o, \Delta\varphi) < \delta_1$,

then

(31.18) $q_1(s)$ and $\bar{q}(s)$ are in U for $s_a \leqq s \leqq s_d$,

and

(31.19) $|q_1^1(s)-q^1(s)|$ and $|\bar{q}^1(s)-q^1(s)|$ are less than $\inf(-q^1)/2$ for $s_d \leqq s \leqq t_2$.

Henceforth we assume that (31.17) is satisfied.

The trajectories T_1 and $\bar{T}$ are identical for $t_1 \leqq s \leqq s_a$ since $\bar{T}$ is obtained from T_1 by disturbances $(0, \Delta_2\varphi)$ which are zero for $s < s_a$. If (31.17) holds, so does (31.18), and consequently, by (31.6), with $q_1(s)$ replacing $q(s)$, we have

(31.20) $$|q_1(s_d) - \bar{q}(s_d) - \eta_2(s_d)|$$
$$\leqq K_4 (s_d-s_a) N_o(0, \Delta_2\varphi) N_{1s}(0, \Delta_2\varphi),$$

in which $\eta_2(s) = \eta(s; q_o, \varphi; 0, \Delta_2\varphi)$.

Moreover in McShane's deduction ([McS], § 6) of (31.3), when $\Delta F \equiv 0$ the factor $N_{1s}(\Delta q_o, \Delta\varphi)$ in the right member of that inequality is necessary only because of the presence of the factor

(31.21) $$\int_0^1 \Delta\varphi^\alpha_{qj}(s, q + \tau\Delta q)\, d\tau$$

in the right member of his equation for $d(\mathcal{C}^i(s))/d\tau$. In our application of (31.6) above $q_1(s)$ replaced $q(s)$; so in (31.21) the integrand is $\Delta_2 \varphi^\alpha_q j(s, q_1 + (\bar{q} - q_1))$, which for $0 \leqq \tau \leqq 1$ and $s_a \leqq s \leqq s_d$ does not exceed a constant multiple of $N^k_{2y}(\Delta q_o, \Delta\varphi)$, and is identically zero both when $s_1 \leqq s < s_a$ and when $s_d < s \leqq s_2$. Consequently we may replace $N_{1s}(0, \Delta_2\varphi)$ by $N^k_{2y}(\Delta q_o, \Delta\varphi)$ in (31.20). But then, from (31.12) and the fact that $\eta_2(s) \equiv 0$ since $\Delta_1\varphi \equiv \Delta\varphi$ on the original trajectory, we see that (31.20) can be replaced by

$$(31.22) \quad |q_1(s_d)-\bar{q}(s_d)| \leqq 2K_4(s_d-s_a)\, N_o(\Delta q_o, \Delta\varphi) N^k_{2y}(\Delta q_o, \Delta\varphi),$$

using (31.11) and (31.13). It then follows from (31.7) and (31.22) that

$$(31.23) \quad |q_1(s_d)-q(s_d)| \leqq (\epsilon/2K)\, N_o(\Delta q_o, \Delta\varphi)\, N^k_{2y}(\Delta q_o, \Delta\varphi).$$

If (31.17) holds, then so does (31.19), and we can apply lemma (28.3) with $\bar{q}$ in place of q_2 and s_d in place of s_3. From this and (31.23) we find that the inequality

$$(31.24) \quad |q_1(s)-\bar{q}(s)| \leqq (\epsilon/2)\, N_o(\Delta q_o, \Delta\varphi)\, N^k_{2y}(\Delta q_o, \Delta\varphi)$$

is valid for $s_d \leqq s \leqq t_2$, and, in particular, for $t_3 \leqq s \leqq t_2$.

By (15.10) there exists a number K_3 such that

$$(31.25)\quad |q_1(s)-q(s)-\eta(s)-(1/2)\zeta(s)| \leqq K_3\, N_{1s}(\Delta q_0, \Delta_1\varphi)\, [N_{2s}(\Delta q_0, \Delta_1\varphi)]^{\varepsilon},$$

$t_3 \leqq s \leqq t_2$, in which $\eta(s)$ is either member of (31.9) and $\zeta(s)$ is either member of (31.10).

We now define

$$(31.26)\quad \delta(\epsilon) = \min\,[\delta_1, \epsilon/2K_3].$$

Then, if

$$(31.27)\quad N_{2y}(\Delta q_0, \Delta\varphi) < \delta(\epsilon),$$

it follows, by (31.16), (31.26), (31.25), and (31.15), that

$$(31.28)\quad |q_1(s)-q(s)-\eta(s)-(1/2)\zeta(s)| \leqq (\epsilon/2)\, N_{1y}(\Delta q_0, \Delta\varphi)\, N_{2y}(\Delta q_0, \Delta\varphi),$$

$t_3 \leqq s \leqq t_2$. But if (31.27) is valid, so is (31.17), by virtue of (31.26) and the inequality $N_0 \leqq N_{1y} \leqq N_{2y}$. Hence (31.24) also holds, and from (31.28) and (31.24) it follows that

$$(31.29)\quad |\bar{q}(s)-q(s)-\eta(s)-(1/2\,\zeta(s)| \leqq \epsilon\, N_{1y}(\Delta q_0, \Delta\varphi) N_{2y}^{k}(\Delta q_0, \Delta\varphi)$$

whenever (31.27) is satisfied. If we now replace s by $s(y)$, $y(t_2) \leqq y \leqq y(t_3)$, we have (31.2). This completes the proof of our theorem.

32. A Disturbance Which Has No Second Differential Effect Relative to the Norm N_{2y}.

In this section we cite an example to show the impossibility of proving the existence of a second differential of the mapping (6.3) relative to the norm N_{2y}.

We shall assume an original trajectory such that air-density ρ and air-temperature θ are constant, E is positive except when air-speed u is zero, $w_x = w_z = 0$, and with initial values $x_0 = y_0 = z_0 = v_{x0} = v_{z0} = 0$, $v_{y0} > 0$. Let Y be the summital height on this original trajectory. Let $\bar{\Delta}$ be a disturbance consisting only of the following abnormal conditions:

$$\Delta y_0 = r, \quad \text{for each real } r;$$

$$\Delta w_x = \begin{cases} 0, & y \leqq Y \\ y - Y, & Y \leqq y \leqq Y + 2r \\ 2r, & Y + 2r \leqq y \end{cases} .$$

For $\Delta y_0 > 0$, if s_a and s_d are the times of passing altitude Y on the ascending and descending branches of the disturbed trajectory, we have

$$s_d - s_a = (8\,\Delta y_0/g)^{1/2} \qquad (32.1)$$
$$+ \text{(infinitesimal of higher order)} ,$$

Near the summit on the disturbed trajectory v_x is near 0, u_x is nearly $-r$, E is approximately equal to its value

E_o at the summit, and $\dot{v}_x$ is therefore nearly $E_o r$. Hence at s_d we have

$$(32.2) \quad v_x(s_d) = E_o\, r\, (s_d - s_a) + \text{(infinitesimal of higher order)}.$$

It follows from theorem (29.2) that, at $y = 0$ on the descending branch of the disturbed trajectory, x is nearly proportional to $v_x(s_d)$. So for $\Delta y_o > 0$ we have, combining (32.1) and (32.2)

$$(32.3) \quad \Delta X \doteq (\text{constant}).\, r^{3/2},$$

the constant factor being non-zero. But since $N_{2y}(\bar{\Delta}) = |2r|$, (32.3) is equivalent to

$$(32.4) \quad \Delta X \doteq (\text{constant})\, [N_{2y}(\bar{\Delta})]^{3/2},$$

the new constant factor also being non-zero.

In case $\Delta y_o \leqq 0$, however, the summital height on the disturbed trajectory is $Y + \Delta y_o$, because $\Delta w_x = 0$ for $y \leqq Y$. It follows that in this case $\Delta x = 0$.

Hence $dX \equiv dx\, (\bar{\Delta} \mid y = 0) \equiv 0$ and $d^2X \equiv d^2x\, (\bar{\Delta} \mid y = 0) \equiv 0$. But then we see from (32.4) that

$$(32.5) \quad |\Delta X - dX - \tfrac{1}{2} d^2X| \doteq K\, [N_{2y}(\bar{\Delta})]^{3/2}$$

in which K is a non-zero constant. The ratio of the left member of (32.5) to $[N_{2y}(\bar{\Delta})]^2$ therefore does not tend

to zero with $N_{2y}(\bar{\Delta})$, so the second differential relative to this norm does not exist.